INFINITE SEQUENCES OF PALINDROMIC TRIANGULAR NUMBERS

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A triangular number, $\Delta(n) = n(n+1)/2$, is palindromic if it is identical with its reverse. It has been established that an infinity of palindromic triangular numbers exists in bases three [1], five [2], and nine [5]. Also, it has been shown [3] that, in a system of numeration with base $(2k+1)^2$, when $k(k+1)/2$ is annexed to $n(n+1)/2$ then

$$[n(n+1)/2](2k+1)^2 + k(k+1)/2 = [(2k+1)n + k][(2k+1)n + k + 1]/2 ,$$

another triangular number. If the first value of n is k, then an infinite sequence of triangular numbers can be generated, each consisting of like "digits," $k(k+1)/2$, so that each member of the sequence is palindromic.

In the following discussion, n and $\Delta(n)$ are expressed in the announced base. An abbreviated notation is employed, wherein a subscript in the decimal system following an expression indicates the number of times it is repeated in the integer containing it. Thus, the repdigit $333333 = 3_6$, $21111000 = 21_4 0_3$, and $1010101 = (10)_3 1$.

The base $(2k+1)^2 = 8[k(k+1)/2] + 1$ is of the form $8m+1$, where m itself is a triangular number. It is not necessary to restrict m to this extent. In general, if n has the form $(10^k - 1)/2$, then $\Delta(n) = (10^{2k} - 1)/2^3$. It follows that in any system of notation with a base, $b = 8m + 1$, a palindromic $\Delta(n) = m_{2k}$ corresponds to the palindromic $n = \overline{4m_k}$.

BASE NINE

The smallest base of the form $8m+1$ is nine, for $m = 1$. Hence $n = 4_k$ generates the palindromic $\Delta(n) = 1_{2k}$, $k = 1, 2, 3, \cdots$. Nine also is of the form $(2k+1)^2$. The above argument regarding the existence of an infinity of palindromic triangular numbers in bases of this type does not deal with the nature of the corresponding n's.

In base nine, for $k = 0, 1, 2, \cdots$, $n = 14_k$ may also be written as

$$n = 10^k + (10^k - 1)/2 = [3(10^k) - 1]/2 = (10^{k+1} - 3)/6 .$$

Then

$$\Delta(n) = (10^{k+1} - 3)(10^{k+1} + 3)/2(6^2) = (10^{2k+2} - 10)/80 = (10^{2k+1} - 1)/8 = 1_{2k+1} .$$

These two results reestablish that, in the scale of nine, any repunit, 1_p, with $p = 1, 2, 3, \cdots$, is a palindromic triangular number.

Furthermore, for $k = 0, 1, 2, \cdots$, we have

$$n = 24_k6 = 2(10^{k+1}) + (10^k - 1)(10)/2 + 6 = [5(10^{k+1}) + 3]/2 .$$

It follows that

$$\begin{aligned}\Delta(n) &= [5(10^{k+1}) + 3][5(10^{k+1}) + 5]/8 \\ &= 5^2(10^{2k+2})/8 + 8(5)(10^{k+1})/8 + 3(5)/8 \\ &= 3(10^{2k+2}) + 10^{k+2}(10^k - 1)/8 + 6(10^{k+1}) + 10(10^k - 1)/8 + 3 \\ &= 31_k61_k3 .\end{aligned}$$

Thus there are two infinite sequences of palindromic triangular numbers in base nine. These do not include all the palindromic $\Delta(n)$ for $n < 42161$. Also, there are:

$\Delta(2) = 3$, $\Delta(3) = 6$, $\Delta(35) = 646$, $\Delta(115) = 6226$, $\Delta(177) = 16661$, $\Delta(353) = 64246$

(the distinct digits are consecutive even digits),

$$\Delta(1387) = 1032301$$

(the distinct digits are consecutive),

$\Delta(1427) = 1075701$, $\Delta(2662) = 3678763$, $\Delta(3525) = 6382836$, $\Delta(3535) = 6428246$

(the distinct digits are consecutive even digits),

$\Delta(4327) = 10477401$, $\Delta(17817) = 167888761$, $\Delta(24286) = 306272603$, $\Delta(24642) = 316070613$, $\Delta(26426) = 362525263$, $\Delta(36055) = 666707666$.

BASES OF FORM 2m + 1

In bases of the form $2m + 1$, if $n = m_k = (10^k - 1)/2$, then

$$\Delta(n) = (10^{2k} - 1)/2^3 = (\overline{2m}_{2k})/2^3 .$$

Now, if

$$[2m(2m + 1) + 2m]/2^3 = m(m + 1)/2 < 2m + 1 ,$$

then $\Delta(n)$ is palindromic. Thus, in base three, $\Delta(1_k) = \overline{01}_k$. In base 5, $\Delta(2_k) = \overline{03}_k$. In base seven, $\Delta(3_k) = \overline{06}_k$. In base nine, $\Delta(4_k) = \overline{11}_k$. In base eleven, $\Delta(5_k) = \overline{14}_k$.

That is, in every odd base not of the form $8m + 1$ there is an infinity of triangular numbers that are smoothly undulating (composed of two alternating unlike digits). In these odd bases < nine, these triangular numbers are palindromic with $2k - 1$ digits. In such odd bases > nine, these triangular numbers consist of **repeated pairs of unlike digits**, so they are not palindromic.

In bases of the form $8m + 1$ (including nine), these triangular numbers are repdigits with $2k$ digits, and are palindromic.

In base three, all of the palindromic triangular numbers for $n < 11(10^4)$ are of the $\Delta(1_1) = \overline{01}_k$ type.

In base five, for $n < 102140$, the other palindromic triangular numbers are

$$\Delta(1) = 1, \quad \Delta(3) = 11, \quad \Delta(13) = 121, \quad \Delta(102) = 3003,$$
$$\Delta(1303) = 1130311, \quad \Delta(1331) = 1222221, \quad \Delta(10232) = 30133103,$$
$$\Delta(12143) = 102121201, \quad \Delta(12243) = 103343301, \quad \Delta(31301) = 1022442201.$$

In base seven, for $n < 54145$, the other palindromic numbers are:

$$\Delta(1) = 1, \quad \Delta(2) = 3, \quad \Delta(15) = 141, \quad \Delta(24) = 333, \quad \Delta(135) = 11211,$$
$$\Delta(242) = 33033, \quad \Delta(254) = 36363, \quad \Delta(1301) = 1012101,$$
$$\Delta(1611) = 1525251, \quad \Delta(2414) = 3251523, \quad \Delta(2424) = 3306033,$$
$$\Delta(2442) = 3352533, \quad \Delta(2522) = 3546453, \quad \Delta(12665) = 100646001,$$
$$\Delta(13065) = 102252201, \quad \Delta(13531) = 112050211, \quad \Delta(15415) = 142323241,$$
$$\Delta(16055) = 15202051, \quad \Delta(23462) = 312444213,$$
$$\Delta(24014) = 321414123, \quad \Delta(25412) = 363030363.$$

Thus, in bases five, seven, and nine (but evidently not in base three) there are palindromic $\Delta(n)$ for which n is palindromic and palindromic $\Delta(n)$ for which n is nonpalindromic.

BASE TWO

In base two, for $k > 1$, if $n = 10^k + 1$, then

$$\begin{aligned}\Delta(n) &= (10^k + 1)(10^k + 10)/10 = (10^k + 1)(10^{k-1} + 1) \\ &= 10^{2k-1} + 10^k + 10^{k-1} + 1 = 10^{2k-1} + 11(10^{k-1}) + 1 \\ &= 10_{k-2}110_{k-2}1 .\end{aligned}$$

For $n < 101101$, in the binary system, palindromic $\Delta(n)$ not contained in this infinite sequence are:

$$\Delta(1) = 1, \quad \Delta(10) = 11, \quad \Delta(110) = 10101, \quad \Delta(10101) = 11100111,$$

$$\Delta(11001) = 101000101, \quad \Delta(101010) = 1110000111 .$$

No infinite sequence of palindromic triangular numbers has been found in base ten [4] or in other even bases > two.

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A NOTE ON THE FERMAT - PELLIAN EQUATION $x^2 - 2y^2 = 1$

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It is a well known fact that $3 + 2\sqrt{2}$ is the fundamental solution of the Fermat-Pellian equation $x^2 - 2y^2 = 1$. Hence, if $u + v\sqrt{2}$ is any other solution then there exists an integer n such that $u + v\sqrt{2} = (3 + 2\sqrt{2})^n$. Let $T = (a_{ij})$ be the 3-by-3 matrix where $a_{12} = a_{21} = 1$, $a_{33} = 3$, and $a_{ij} = 2$ for all other values. It is interesting to observe that there exists a relationship between the integral powers of T and $3 + 2\sqrt{2}$. In fact, a necessary and sufficient condition for $M = T^n$ is that $M = (b_{ij})$ with $b_{33} = 2m + 1$, $b_{12} = b_{21} = m$, $b_{11} = b_{22} = m + 1$ and $b_{13} = b_{23} = b_{31} = b_{32} = v$, where $(2m + 1)^2 - 2v^2 = 1$. If $n \geq 0$ both the necessary and sufficient condition follow by induction. Using this fact, it then follows for $n < 0$.