AN EXTENSION OF FIBONACCI'S SEQUENCE

P. J. deBRUIJN
Zoutkeetlaan 1, Oegstgeest, Holland

Fibonacci's sequence is generally known as the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$ defined by $u_1 = 1$, $u_2 = 1$, $u_{n+1} = u_n + u_{n-1}$, in which n is a positive integer $\geqslant 2$. It is easy to extend this sequence in such a way that n may be any integer number.

We then get:

$$\begin{array}{cccccccccccccccccc} \cdots & -21, & 13, & -8, & 5, & -3, & 2, & -1, & 1, & 0, & 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & u_{-8} & u_{-7} & u_{-6} & u_{-5} & u_{-4} & u_{-3} & u_{-2} & u_{-1} & u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{array}$$

In this sequence we have:

(1a) $$u_1 = 1, \quad u_2 = 1, \quad u_{n+1} = u_n + u_{n-1} \quad \text{for all} \quad n \in Z.$$

The following definition is known to be equivalent to the previous one:

(1b) $$u_n = \frac{a^n - \beta^n}{a - \beta} \quad \text{for all} \quad n \in Z,$$

in which a is the positive root and β the negative root of the equation $x^2 = x + 1$.

We know the following relations involving a and β to be valid:

$$a = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} = 1.6180339 \cdots$$

$$\beta = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5} = -0.6180339 \cdots$$

$$a^2 = a + 1, \quad \beta^2 = \beta + 1, \quad a\beta = -1, \quad a + \beta = 1, \quad a - \beta = \sqrt{5}\,.$$

The proof of the identities in this paper will in most cases be based upon $a^2 = a + 1$.

The purpose of this article is to study the results of an extension of definition (1b) in such a way that for n not only integers, but also rational numbers, and even all real numbers can be chosen.

If we try $n = \tfrac{1}{2}$ in definition (1b), we get

$$u_{1/2} = \frac{a^{1/2} - \beta^{1/2}}{a - \beta},$$

in which $\beta^{1/2} = \sqrt{\beta}$ causes trouble, because β is negative.

To avoid these difficulties, we define:

(2) $$u_n = \frac{a^{2n} - \cos n\pi + i \sin n\pi}{(a - \beta)a^n},$$

or $u_n = x_n + iy_n$, in which

$$x_n = \frac{a^{2n} - \cos n\pi}{(a - \beta)a^n} \quad \text{and} \quad y_n = \frac{\sin n\pi}{(a - \beta)a^n}.$$

In this definition we have: $n \in R$, $u_n \in C$.

First we shall have to show, of course, that this definition is equivalent to (1b) for $n \in Z$. We calculate:

$$u_1 = \frac{a^2 - \cos\pi + i\sin\pi}{(a-\beta)a} = \frac{a^2+1}{a^2-a\beta} = \frac{a^2+1}{a^2+1} = 1,$$

$$u_2 = \frac{a^4 - \cos 2\pi + i\sin 2\pi}{(a-\beta)a^2} = \frac{a^4-1}{(a-\beta)a^2} = \frac{(a^2+1)(a^2-1)}{(a-\beta)a^2} = \frac{(a^2+1)a}{(a-\beta)a^2} = \frac{a^2+1}{a^2-a\beta} = 1.$$

Now we will show that for all n the relation $u_{n+1} = u_n + u_{n-1}$ remains valid.

$$u_{n+1} = \frac{a^{2n+2} - \cos(n+1)\pi + i\sin(n+1)\pi}{(a-\beta)a^{n+1}} = \frac{a^{2n+2} + \cos n\pi - i\sin n\pi}{(a-\beta)a^{n+1}},$$

$$u_{n-1} = \frac{a^{2n-2} - \cos(n-1)\pi + i\sin(n-1)\pi}{(a-\beta)a^{n-1}} = \frac{a^{2n-2} + \cos n\pi - i\sin n\pi}{(a-\beta)a^{n-1}}.$$

The identity which we have to prove can now be reduced to:

$$a^{2n+2} + \cos n\pi - i\sin n\pi = a^{2n+1} - a\cos n\pi + ai\sin n\pi + a^{2n} + a^2\cos n\pi - a^2 i\sin n\pi,$$

or:

$$(a^2 - a - 1)(a^{2n} - \cos n\pi + i\sin n\pi) = 0,$$

which is a proper identity, since $a^2 - a - 1 = 0$.

The numbers, introduced by definition (2) also satisfy identically the relation $u_m u_n + u_{m+1}u_{n+1} = u_{m+n+1}$, which is well known for the ordinary Fibonacci numbers. The truth of this assertion can also be verified without too much difficulty.

Furthermore we can show that for the moduli of the complex numbers the relation $|u_{-n}| = |u_n|$ is valid, just as for the real numbers. For $x_{-n}^2 + y_{-n}^2 = x_n^2 + y_n^2$ is equivalent to

$$\left(\frac{a^{-2n} - \cos n\pi}{(a-\beta)a^{-n}}\right)^2 + \left(\frac{\sin n\pi}{(a-\beta)a^{-n}}\right)^2 = \left(\frac{a^{2n} - \cos n\pi}{(a-\beta)a^{n}}\right)^2 + \left(\frac{\sin n\pi}{(a-\beta)a^{n}}\right)^2,$$

and this in its turn is identical to:

$$\frac{a^{-4n} - 2a^{-2n}\cos n\pi + 1}{(a-\beta)^2 a^{-2n}} = \frac{a^{4n} - 2a^{2n}\cos n\pi + 1}{(a-\beta)^2 a^{2n}},$$

or:

$$a^{-2n} - 2\cos n\pi + a^{2n} = a^{2n} - 2\cos n\pi + a^{-2n} \quad \text{q.e.d.}$$

We now calculate the numerical values of u_n, for n climbing from –4 to +4, with intervals of 1/6 as shown in Table 1.

If we take a close look at these numbers, we find that

$$u_{1/2} = iu_{-1/2} = 0.569 + 0.352i,$$

$$u_{-1\frac{1}{2}} = iu_{1\frac{1}{2}} = 0.217 + 0.921i,$$

$$u_{2\frac{1}{2}} = iu_{-2\frac{1}{2}} = 1.489 + 0.134i,$$

etc., etc.

It is simple to prove this property from definition (2), and it is clear that it corresponds with $|u_{-n}| = |u_n|$.

If we make a map of the newly introduced numbers in the complex plane, we get the interesting picture shown in Fig. 1. The curve that we have thus found intersects the x-axis in those real points corresponding with the well-known Fibonacci numbers for $n \in Z$.

For decreasing negative values of n it has the shape of a spiral, and for increasing positive values of n it has the shape of a "sinus-like" curve, with increasing "wave-length" and decreasing "amplitude."

Note how the relation $|u_{-n}| = |u_n|$ is made visible through this graphical representation of u_n.

On differentiating,

$$x_n = \frac{\alpha^{2n} - \cos n\pi}{(\alpha-\beta)\alpha^n}, \qquad y_n = \frac{\sin n\pi}{(\alpha-\beta)\alpha^n}$$

with n as independent variable, we find:

$$\frac{dx_n}{dn} = \frac{\ln \alpha(\alpha^{2n} + \cos n\pi) + \pi \sin n\pi}{(\alpha-\beta)\alpha^n},$$

$$\frac{dy_n}{dn} = \frac{\pi \cos n\pi - \ln \alpha \sin n\pi}{(\alpha-\beta)\alpha^n},$$

so that

$$\frac{dy_n}{dx_n} = \frac{\pi \cos n\pi - \ln \alpha \sin n\pi}{\ln \alpha(\alpha^{2n} + \cos n\pi) + \pi \sin n\pi}.$$

For instance:

$$\frac{dy}{dx_{n=0}} = \frac{\pi}{2 \ln \alpha} = \frac{\pi \log e}{2 \log \alpha} = \frac{3.1416 \times 0.4343}{2 \times 0.2090} = 3.264.$$

$$\frac{dy}{dx_{n=1}} = -\frac{\pi}{\alpha \ln \alpha} = -\frac{\pi \log e}{\alpha \log \alpha}$$

$$= -\frac{3.1416 \times 0.4343}{1.618 \times 0.2090} = -4.035.$$

$$\frac{dy}{dx_{n=-1}} = \frac{\pi\alpha}{\ln \alpha} = \frac{\pi\alpha \log e}{\log \alpha}$$

$$= \frac{3.1416 \times 1.618 \times 0.4343}{0.2090} = 10.56$$

etc., etc.

Among the points in which the curve intersects itself, there is one with $y \neq 0$, a complex number z, so that $z \in C$ but $z \notin R$. With the extension we now have achieved, we can make a similar extension for all Fibonacci-like sequences

If we start with any two complex numbers, say z_1 and z_2, adding them to find the following number we get

Table 1

$6n$	$u_n = x_n + iy_n$	$6n$	$u_n = x_n + iy_n$
⋮			
−24	−3.000 + 0.000 i	0	0.000 + 0.000 i
−23	−2.380 + 1.415 i	+1	+0.127 + 0.206 i
−22	−1.229 + 2.261 i	+2	+0.335 + 0.330 i
−21	+ 0.083 + 2.410 i	+3	+0.569 + 0.352 i
−20	+ 1.203 + 1.926 i	+4	+0.779 + 0.281 i
−19	+ 1.875 + 1.026 i	+5	+0.927 + 0.150 i
−18	+ 2.000 + 0.000 i	+6	+1.000 + 0.000 i
−17	+ 1.629 + 0.874 i	+7	+1.005 − 0.128 i
−16	+ 0.931 − 1.398 i	+8	+0.967 − 0.204 i
−15	+ 0.134 − 1.489 i	+9	+0.920 − 0.217 i
−14	−0.542 − 1.190 i	+10	+0.897 − 0.174 i
−13	−0.941 − 0.634 i	+11	+0.920 − 0.093 i
−12	−1.000 + 0.000 i	+12	+1.000 + 0.000 i
−11	−0.751 + 0.540 i	+13	+1.132 + 0.079 i
−10	−0.298 + 0.864 i	+14	+1.302 + 0.126 i
−9	+ 0.217 + 0.921 i	+15	+1.489 + 0.134 i
−8	+ 0.661 + 0.736 i	+16	+1.676 + 0.107 i
−7	+ 0.934 + 0.392 i	+17	+1.848 + 0.057 i
−6	+ 1.000 + 0.000 i	+18	+2.000 + 0.000 i
−5	+ 0.878 − 0.334 i	+19	+2.137 − 0.049 i
−4	+ 0.633 − 0.534 i	+20	+2.269 − 0.078 i
−3	+ 0.352 − 0.569 i	+21	+2.410 − 0.083 i
−2	+ 0.118 − 0.455 i	+22	+2.573 − 0.066 i
−1	−0.007 − 0.242 i	+23	+2.768 − 0.035 i
0	0.000 + 0.000 i	+24	+3.000 + 0.000 i
$6n$	$u_n = x_n + iy_n$		⋮

$$z_1,\ z_2,\ z_1 + z_2,\ z_1 + 2z_2,\ 2z_1 + 3z_2,\ 3z_1 + 5z_2,\ 5z_1 + 8z_2,\ 8z_1 + 13z_2,$$

etc., etc. The coefficients are Fibonacci numbers.

To find the extension of this sequence, all we have to do is to apply the extension to the coefficients.

In this manner we will now study the sequence that appears when we start with $z_1 = 1, z_2 = i$. Then we have:

$$1,\ i,\ 1 + i,\ 1 + 2i,\ 2 + 3i,\ 3 + 5i,\ 5 + 8i,$$

etc. It is clear that we can start by extension "to the left," to find:

$$\cdots,\ 5 - 3i,\ -3 + 2i,\ 2 - i,\ -1 + i,\ 1,\ i,\ 1 + i,\ 1 + 2i,\ 2 + 3i,\ 3 + 5i,\ \cdots.$$

For reasons of symmetry we shall refer to these terms as v_k, in such a way that $v_{-1/2} = 1$ and

$$v_{+1/2} = i,\ v_{+1\frac{1}{2}} = 1 + i,\ v_{+2\frac{1}{2}} = 1 + 2i,\ v_{+3\frac{1}{2}} = 2 + 3i, \cdots$$

$$v_{-1\frac{1}{2}} = i - 1,\ v_{-2\frac{1}{2}} = 2 - i,\ v_{-3\frac{1}{2}} = -3 + 2i, \cdots.$$

The relation between the v-sequence and the u-sequence is: $v_k = u_{k-1/2} + u_{k+1/2}i$. Therefore:

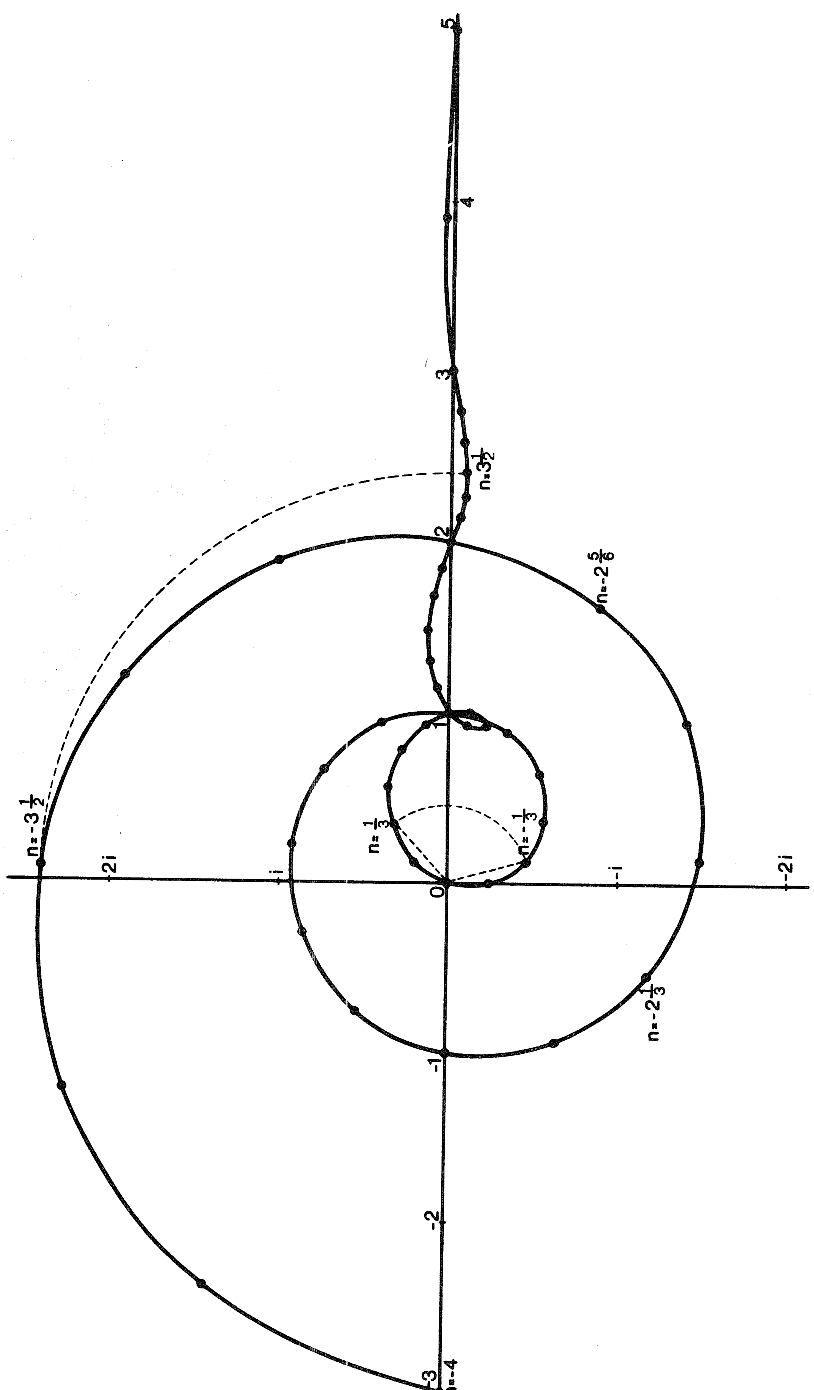

Fig. 1 Graphic Representation of the Complex Numbers of the Extended Fibonacci Sequence, According to Definition (2) for $-4 \leqslant n \leqslant 5$

$$v_k = u_{k-\frac12} + u_{k+\frac12}i = (x_{k-\frac12} + iy_{k-\frac12}) + (x_{k+\frac12} + iy_{k+\frac12})i = (x_{k-\frac12} - y_{k+\frac12}) + i(y_{k-\frac12} + x_{k+\frac12}).$$

We shall now demonstrate that $|v_{-k}| = |v_k|$.

$$|v_k|^2 = (x_{k-\frac12} - y_{k+\frac12})^2 + (y_{k-\frac12} + x_{k+\frac12})^2 = (x^2_{k-\frac12} + y^2_{k-\frac12}) + (x^2_{k+\frac12} + y^2_{k+\frac12}) - 2(x_{k-\frac12}y_{k+\frac12} - y_{k-\frac12}x_{k+\frac12}).$$

We can now say that:

$$|v_k|^2 = |u_{k-\frac12}|^2 + |u_{k+\frac12}|^2 - 2(x_{k-\frac12}y_{k+\frac12} - y_{k-\frac12}x_{k+\frac12}).$$

Therefore:

$$|v_{-k}|^2 = |u_{-k-\frac12}|^2 + |u_{-k+\frac12}|^2 - 2(x_{-k-\frac12}y_{-k+\frac12} - y_{-k-\frac12}x_{-k+\frac12}) = |u_{k+\frac12}|^2 + |u_{k-\frac12}|^2 - 2(x_{-k-\frac12}y_{-k+\frac12} - y_{-k-\frac12}x_{-k+\frac12}).$$

so that the relation that we want to prove, namely $|v_{-k}| = |v_k|$, or $|v_{-k}|^2 = |v_k|^2$, is equivalent to

$$x_{k-\frac12}y_{k+\frac12} - y_{k-\frac12}x_{k+\frac12} = x_{-k-\frac12}y_{-k+\frac12} - y_{-k-\frac12}x_{-k+\frac12}.$$

When we now proceed to introduce the index t by means of $k = t + \frac12;\ -k = -t - \frac12$, we have to prove that:

$$x_t y_{t+1} - y_t x_{t+1} = x_{-t-1}y_{-t} - y_{-t-1}x_{-t}.$$

Or:

$$\frac{a^{2t} - \cos t\pi}{(a-\beta)a^t} \times \frac{\sin(t+1)\pi}{(a-\beta)a^{t+1}} - \frac{\sin t\pi}{(a-\beta)a^t} \times \frac{a^{2(t+1)} - \cos(t+1)\pi}{(a-\beta)a^{t+1}}$$

$$= \frac{a^{2(-t-1)} - \cos(-t-1)\pi}{(a-\beta)a^{-t-1}} \times \frac{\sin(-t)\pi}{(a-\beta)a^{-t}} - \frac{\sin(-t-1)\pi}{(a-\beta)a^{-t-1}} \times \frac{a^{2(-t)} - \cos(-t)\pi}{(a-\beta)a^{-t}}.$$

This is an identity, if completely worked out.

We have already seen that if $v_k = a_k + ib_k$, then $a_k = x_{k-\frac12} - y_{k+\frac12}$ and $b_k = y_{k-\frac12} + x_{k+\frac12}$. Thus:

$$a_k = x_{k-\frac12} - y_{k+\frac12} = \frac{a^{2k-1} - \cos(k\pi - \frac12\pi)}{a^{k-\frac12}(a-\beta)} - \frac{\sin(k\pi + \frac12\pi)}{(a-\beta)a^{k+\frac12}}$$

Or:

$$a_k = \frac{a^{2k} - a\sin k\pi - \cos k\pi}{(a-\beta)a^{k+\frac12}}.$$

In the same way we derive from $b_k = y_{k-\frac12} + x_{k+\frac12}$:

$$b_k = \frac{a^{2k+1} - a\cos k\pi + \sin k\pi}{(a-\beta)a^{k+\frac12}}.$$

It is now fairly easy to calculate some values of v_k, simply by choosing different values of k; we find

$$v_{\frac12} = i, \quad v_{1\frac12} = 1 + i, \quad v_{2\frac12} = 1 + 2i,$$

as it should be. We also have:

$$v_1 = \frac{1}{\sqrt{a}} + i\sqrt{a}, \quad v_{-1} = \frac{1}{\sqrt{a}} + i\sqrt{a},$$

(so that $v_{-1} = v_1$), and $v_0 = 0$. Also

$$v_2 = \frac{1}{\sqrt{a}} + i\sqrt{a}\ (= v_{-1} = v_1) \text{ and } v_{-2} = -\frac{1}{\sqrt{a}} - i\sqrt{a};\ v_3 = \frac{2}{\sqrt{a}} + 2i\sqrt{a} \text{ and } v_{-3} = \frac{2}{\sqrt{a}} + 2i\sqrt{a},\ v_4 = \frac{3}{\sqrt{a}} + 3i\sqrt{a}.$$

It now seems very likely that

$$v_k = \left(\frac{1}{\sqrt{a}} + i\sqrt{a}\right)u_k,$$

for all values of k. Indeed we have:

$$(a^{-\frac12} + ia^{\frac12}) \times u_k = (a^{-\frac12} + ia^{\frac12})(x_k + iy_k) = (a^{-\frac12}x_k - a^{\frac12}y_k) + i(a^{-\frac12}y_k + a^{\frac12}x_k),$$

whereas

$$a^{-1/2}x_k - a^{1/2}y_k = \frac{a^{-1/2}(a^{2k} - \cos k\pi)}{(a-\beta)a^k} - \frac{a^{1/2}\sin k\pi}{(a-\beta)a^k} = \frac{a^{2k} - \cos k\pi - a\sin k\pi}{(a-\beta)a^{k+1/2}} = a_k,$$

and in the same way we prove that $a^{-1/2}y_k + a^{1/2}x_k = b_k$, so that $(a^{-1/2} + ia^{1/2})u_k = a_k + ib_k = v_k$, which had to be proved. The relation

$$v_k = \left(\frac{1}{\sqrt{a}} + i\sqrt{a}\right) u_k$$

implies that the graphic representation of the numbers v_k in the complex plane has the same shape as the one that we have found previously for u_k :

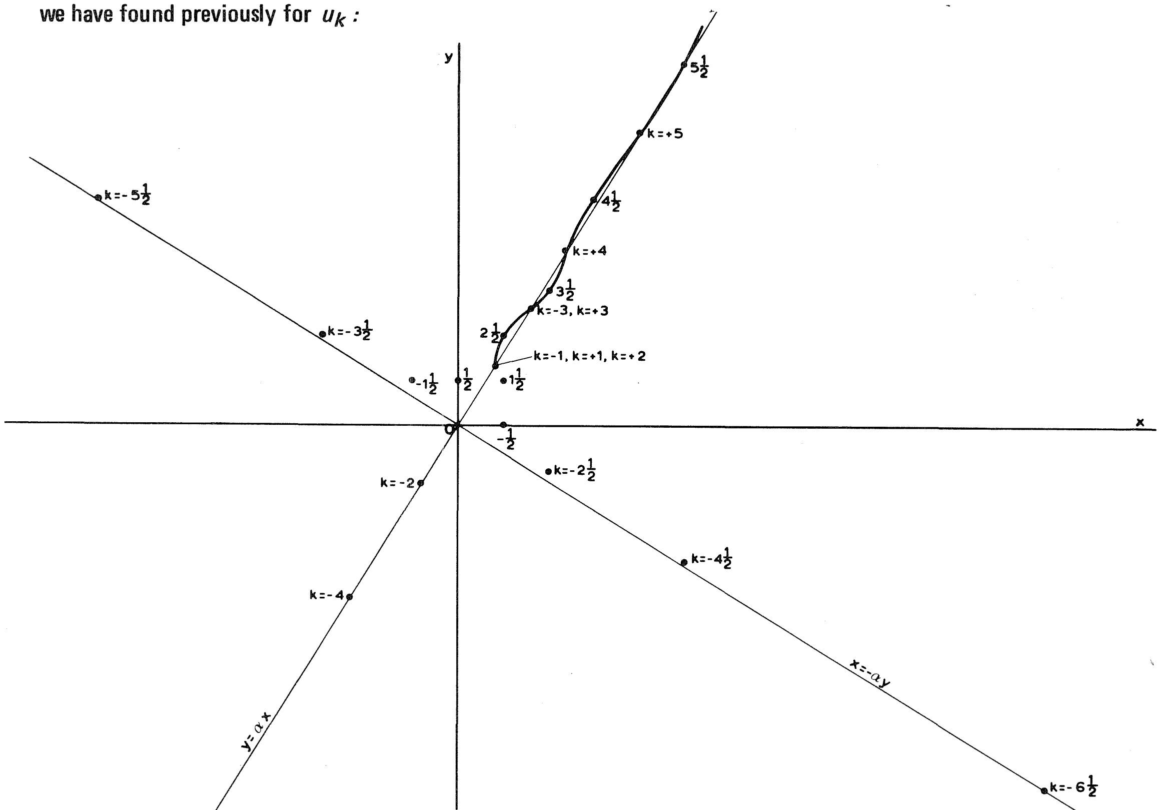

Fig. 2 Graphic Representation of the Numbers v_k in the Complex Plane

There is one continuous curve going through all these points, a curve that originates from the one in Fig. 1 by multiplication with

$$\frac{1}{\sqrt{a}} + i\sqrt{a} .$$

It is clearly shown how the points $(0,1)$; $(1,1)$; $(1,2)$; $(2,3)$; $(3,5)$; $(5,8)$; $(8,13)$; ··· belonging to the index-values ½, 1½, 2½, 3½, 4½, 5½, 6½, ··· of k are lying closer to the asymptote $y = ax$ as k increases, thus indicating that

$$\lim_{k\to\infty} \frac{u_{k+1}}{u_k} = a .$$

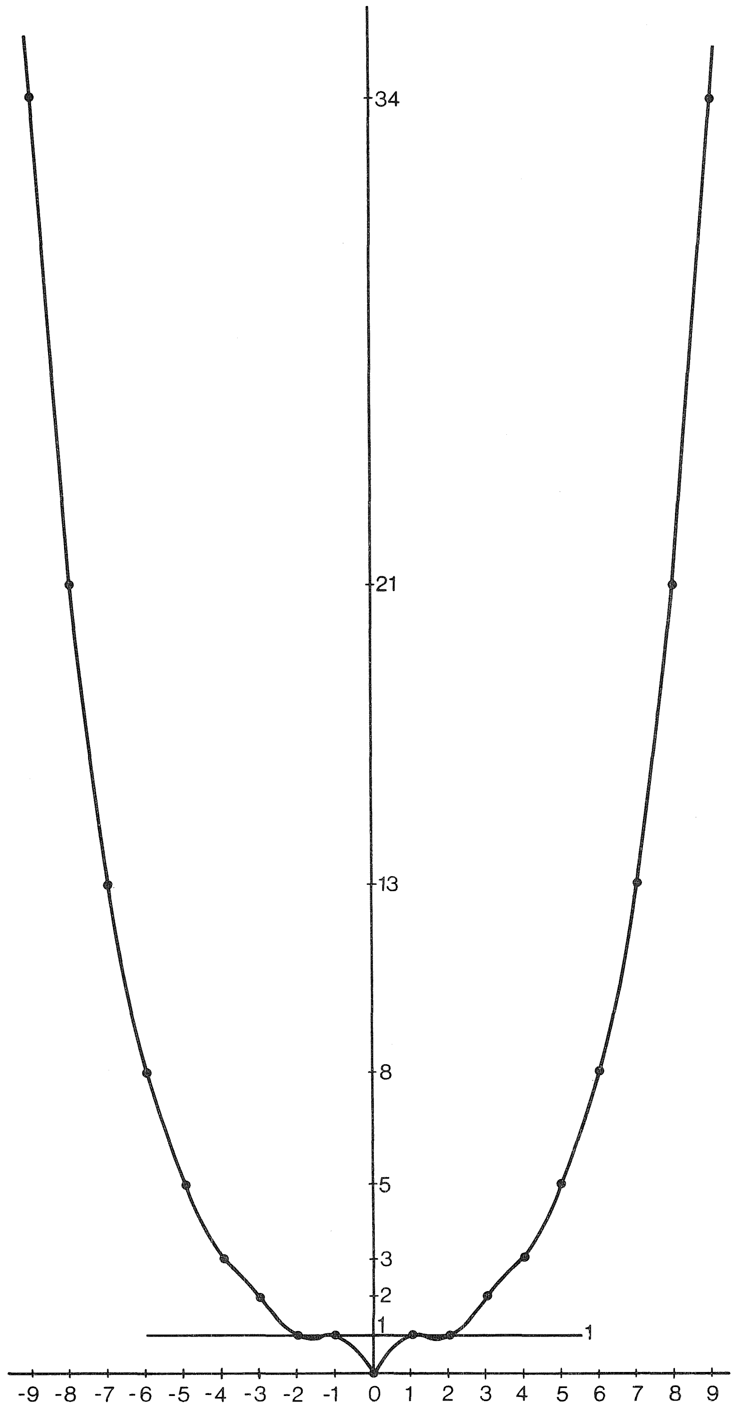

Fig. 3 Graph of $|u_n|$ as a function of n

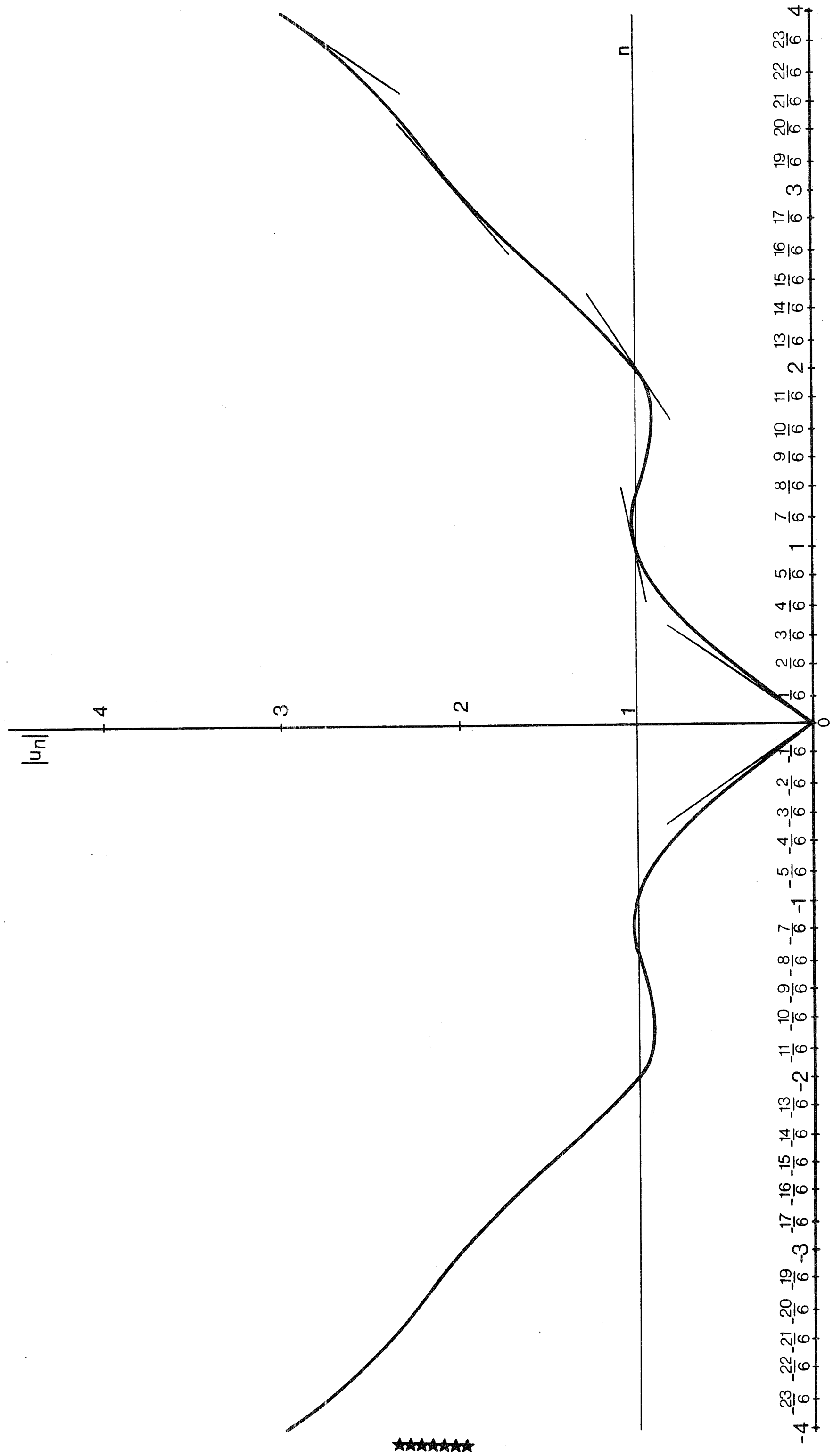

Fig. 4 Fig. 3 Enlarged to Show the Behavior of $|u_n|$ for $n = 0$.