

EXPLICIT EXPRESSIONS FOR POWERS OF LINEAR RECURSIVE SEQUENCES

A. G. SHANNON

The New South Wales Institute of Technology, Broadway, N.S.W., Australia

1. DEFINITIONS

Van der Poorten [6] in a generalization of a result of Shannon and Horadam [8] has shown that (in my notation) if $\{w_n^{(i)}\}$ is a linear recursive sequence of arbitrary order i defined by the recurrence relation

$$(1.1) \quad w_n^{(i)} = \sum_{j=1}^i P_{ij} w_{n-j}^{(i)}, \quad n \geq i,$$

where the P_{ij} are arbitrary integers, with suitable initial values $w_0^{(i)}, w_1^{(i)}, \dots, w_{i-1}^{(i)}$, then the sequence of powers $\{w_n^{(i)r}\}$, for integers $r \geq 1$, satisfies a similar recurrence relation of order at most

$$\binom{r+i-1}{r}.$$

In other words, he has established the existence of generating functions

$$(1.2) \quad w_r^{(i)}(x) = \sum_{n=0}^{\infty} w_n^{(i)r} x^n, \quad (w_n^{(i)r} \equiv (w_n^{(i)})^r).$$

The aim here is to find the recurrence relation for $\{w_n^{(i)r}\}$ and an explicit expression for $w_r^{(i)}(x)$. We shall concern ourselves with the non-degenerate case only; the degenerate case is no more difficult because the order of the recurrence relation for $\{w_n^{(i)r}\}$ is then lower than

$$\binom{r+i-1}{r}.$$

It is worth noting in passing that Marshall Hall [1] looked at the divisibility properties of a third-order sequence by a similar approach. From a second-order sequence with auxiliary equation roots a_1 and a_2 he formed a third-order sequence with auxiliary equation roots $a_1^2, a_2^2, a_1 a_2$.

2. RECURRENCE RELATION FOR SEQUENCE OF POWERS

Van der Poorten proved that if the auxiliary equation for $\{w_n^{(i)}\}$ is

$$(2.1) \quad g(x) \equiv x^i - \sum_{j=1}^i P_{ij} x^{i-j} = \prod_{t=1}^i (x - a_{it}) = 0,$$

then the sequence $\{w_n^{(i)r}\}$ satisfies a linear recurrence relation of order

$$\binom{r+i-1}{r}$$

with auxiliary equation

$$(2.2) \quad g_r(x) \equiv \prod_{\sum \lambda_n = r} (x - a_{i1}^{\lambda_1} a_{i2}^{\lambda_2} \dots a_{ii}^{\lambda_i}) = 0,$$

the zeros of which are exactly the zeros of $g(x)$ taken r at a time.

We now set

$$(2.3) \quad g_r(x) = x^u - \sum_{j=1}^u R_{uj} x^{u-j}, \quad u = \binom{r+i-1}{r},$$

and we seek the R_{uj} .

Macmahon [5, p. 3] defines h_j , the homogeneous product sum of weight j of the quantities a_{jr} , as the sum of a number of symmetric functions, each of which is denoted by a partition of the number j . He showed that in our notation

$$h_j = \sum_{\sum n \lambda_n = j} \frac{(\sum \lambda)!}{\lambda_1! \lambda_2! \dots \lambda_j!} P_{i1}^{\lambda_1} P_{i2}^{\lambda_2} \dots P_{ij}^{\lambda_j}.$$

The first three cases of h_j are

$$h_1 = P_{i1} = \sum a_{i1},$$

$$h_2 = P_{i1}^2 + P_{i2} = \sum a_{i1}^2 + \sum a_{i1} a_{i2},$$

$$h_3 = P_{i1}^3 + 2P_{i1}P_{i2} + P_{i3} = \sum a_{i1}^3 + \sum a_{i1}^2 a_{i2} + \sum a_{i1} a_{i2} a_{i3}.$$

Now $g_r(x) = 0$ is the equation whose zeros are the several terms of h_r with $a_{ij} = 0$ for $j > i$, since from its construction its zeros are a_{ij} taken r at a time; that is,

$$R_{u1} = h_r \quad \text{with} \quad a_{ij} = 0 \quad \text{for} \quad j > i,$$

since we have supposed that there are

$$\binom{r+i-1}{r} = u$$

distinct zeros of $g_r(x) = 0$.

Macmahon has proved [5, p. 19] that H_j^r , the homogeneous product sum, j together, of the whole of the terms of h_r , can be represented in terms of the symmetric functions (denoted by []) of the roots of

$$x^i - h_1 x^{i-1} + h_2 x^{i-2} - \dots = 0$$

by

$$(2.4) \quad H_j^r = \sum_{\sum n \mu_n = j} (-1)^{r(3\mu_2 + 5\mu_4 + \dots)} \frac{[1^r]^{\mu_1} [2^r]^{\mu_2} [3^r]^{\mu_3} \dots}{1^{\mu_1} \cdot 2^{\mu_2} \cdot 3^{\mu_3} \dots \mu_1! \mu_2! \mu_3! \dots}$$

Some examples of H_j^r are (with $a_{ij} = 0$ for $j > i$)

$$H_2^1 = a_{21}^2 + a_{22}^2 + a_{21} a_{22},$$

$$H_2^2 = a_{21}^4 + a_{22}^4 + 2a_{21}^2 a_{22}^2 + a_{21}^3 a_{22} + a_{21} a_{22}^3,$$

$$H_2^3 = a_{21}^6 + a_{22}^6 + 2a_{21}^3 a_{22}^3 + a_{21}^5 a_{22} + a_{21} a_{22}^5 + 2a_{21}^2 a_{22}^4 + 2a_{21}^4 a_{22}^2,$$

$$H_2^2 = H_2^1 H_2^1 = a_{21}^4 + a_{22}^4 + 3a_{21}^2 a_{22}^2 + 2a_{21}^3 a_{22} + 2a_{21} a_{22}^3,$$

$$H_2^3 = a_{21}^6 + a_{22}^6 + 7a_{21}^3 a_{22}^3 + 3a_{21}^5 a_{22} + 3a_{21} a_{22}^5 + 6a_{21}^4 a_{22}^2 + 6a_{21}^2 a_{22}^4.$$

h_m is the homogeneous product sum of weight m of the terms of P_{i1} . H_r^m is the homogeneous product sum of weight m of the terms of R_{u1} .

$(-1)^{j+1} P_{ij}$ is the product sum, j together, of the terms of P_{i1} .

$(-1)^{j+1} R_{uj}$ is the product sum, j together, of the terms of R_{u1} . It follows directly from Macmahon [5, p. 4] that

$$P_{ij} = \sum_{\Sigma n \lambda_n = j} (-1)^{1+\Sigma \lambda} \frac{(\Sigma \lambda)!}{\lambda_1! \lambda_2! \cdots \lambda_j!} \prod_{m=1}^j h_m^{\lambda_m} ,$$

and so

$$R_{uj} = \sum_{\Sigma n \lambda_n = j} (-1)^{1+\Sigma \lambda} \frac{(\Sigma \lambda)!}{\lambda_1! \lambda_2! \cdots \lambda_j!} \prod_{m=1}^j H_m^{\lambda_m} .$$

For example,

$$\begin{aligned} R_{31} &= H_2 = a_{21}^2 + a_{22}^2 + a_{21}a_{22} , \\ R_{32} &= -H_2^2 + H_2 = -(\Sigma a_{21}^4 + 2\Sigma a_{21}^3 a_{22} + 3a_{21}^2 a_{22}^2) \\ &\quad + (\Sigma a_{21}^4 + \Sigma a_{21}^3 a_{22} + 2a_{21}^2 a_{22}^2) \\ &= -\Sigma a_{21}^3 a_{22} - \Sigma a_{21}^2 a_{22}^2 , \\ R_{33} &= H_2^3 + H_2 - 2H_2 H_2 = a_{21}^3 a_{22}^3 . \end{aligned}$$

We can verify these results by utilizing some of the properties of the generalized sequence of numbers $\{w_n^{(2)}\}$ developed by Horadam [3].

From Eq. (27) of Horadam's paper we have that

$$(2.5) \quad w_n^{(2)} w_{n-2}^{(2)} - w_{n-1}^{(2)2} = (-P_{22})^{n-2} e ,$$

where

$$e = P_{21} w_0^{(2)} w_1^{(2)} + P_{22} w_0^{(2)2} - w_1^{(2)2} .$$

Thus

$$w_{n-1}^{(2)} w_{n-3}^{(2)} - w_{n-2}^{(2)2} = (-P_{22})^{n-3} e$$

and

$$(2.6) \quad P_{22} w_{n-2}^{(2)2} - P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} = (-P_{22})^{n-2} e .$$

Subtracting (2.5) from (2.6), we get

$$(2.7) \quad P_{22} w_{n-2}^{(2)} + w_{n-1}^{(2)} = P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} + w_n^{(2)} w_{n-2}^{(2)} .$$

But

$$w_n^{(2)} - P_{22} w_{n-2}^{(2)} = P_{21} w_{n-1}^{(2)} ,$$

and

$$w_{n-1}^{(2)} - P_{22} w_{n-3}^{(2)} = P_{21} w_{n-2}^{(2)} ,$$

so

$$w_n^{(2)} + P_{22} w_{n-2}^{(2)2} - 2P_{22} w_n^{(2)} w_{n-2}^{(2)} = P_{21} w_{n-1}^{(2)2} ,$$

and

$$P_{22} w_{n-1}^{(2)2} + P_{22} w_{n-3}^{(2)2} - 2P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} = P_{21}^2 P_{22} w_{n-2}^{(2)2} .$$

Adding the last two equations we obtain

$$w_n^{(2)2} + P_{22} w_{n-1}^{(2)2} + P_{22} w_{n-2}^{(2)2} + P_{22}^3 w_{n-3}^{(2)2} - 2P_{22} (P_{22} w_{n-1}^{(2)} w_{n-3}^{(2)} + w_n^{(2)} w_{n-2}^{(2)}) = P_{21}^2 w_{n-1}^{(2)2} + P_{21}^2 P_{22} w_{n-2}^{(2)2} .$$

Combining this with (2.7) we then have

$$(2.8) \quad w_n^{(2)^2} = (P_{21}^2 + P_{22})w_{n-1}^{(2)^2} + (P_{22}^2 + P_{21}^2 P_{22})w_{n-2}^{(2)^2} + (-P_{22}^3)w_{n-3}^{(2)^2},$$

so

$$\begin{aligned} R_{31} &= P_{21}^2 + P_{22} = a_{21}^2 + a_{22}^2 + a_{21}a_{22}, \\ R_{32} &= P_{22} + P_{21}P_{22} = -a_{21}^3 a_{22} - a_{21}a_{22}^3 - a_{21}^2 a_{22}^2, \\ R_{33} &= -P_{22}^3 = a_{21}^3 a_{22}^3, \end{aligned}$$

as required.

To obtain an expression for H_r in terms of a_{ij} , we now use a result of Macmahon, namely,

$$[u^r] = (-1)^{r(u+1)} \sigma_u,$$

where σ_u denotes the sum of the u^{th} powers of the roots of $g_r(x) = 0$. It is sufficient for our purposes to state that Macmahon has shown that σ_u is the homogeneous product sum of order r of the quantities a_{ij}^u . It is thus given by

$$\sigma_u = \sum_{\Sigma t=r} \prod_m a_{im}^{ut}$$

by analogy with

$$h_r = \sum_{\Sigma t=r} \prod_m a_{im}^t,$$

the homogeneous product sum of order r of the quantities a_{ij} . We now define σ_{iu} , the homogeneous product sum of order r of the quantities a_{ij}^u such that $a_{ij} = 0$ for $j > i$:

$$\sigma_{iu} = \sum_{\Sigma v=r} \prod_{j=1}^i a_{ij}^{uvj},$$

and we introduce the term

$$\sigma_{iur} = (-1)^{r(u+1)} \sigma_{iu}.$$

We have thus established that for

$$(2.9) \quad w_n^{(i)^r} = \sum_{j=1}^u R_{uj} w_{n-j}^{(i)^r},$$

$$R_{uj} = \sum_{\Sigma n \lambda_n = j} (-1)^{1+\Sigma \lambda} \frac{(\Sigma \lambda)!}{\lambda_1! \lambda_2! \dots \lambda_j!} \prod_{m=1}^i H_r^{\lambda_m m},$$

where

$$H_r^m = \sum_{\Sigma n \mu_n = m} (-1)^{r(3\mu_2 + 5\mu_4 + \dots)} \prod_{v=1}^m \frac{(\sigma_{iur})^{\mu_v}}{v^{\mu_v} \cdot \mu_v},$$

and

$$\sigma_{iur} = (-1)^{r(u+1)} \sum_{\Sigma v=r} \prod_{j=1}^i a_{ij}^{uvj},$$

and

$$u = \binom{i+r-1}{r}.$$

It is of interest to note that another formula for σ_{iur} can be given by

$$(2.9) \quad \sigma_{iur} = (-1)^{r(u+1)} \sum_{j=1}^i a_{ij}^{(i+r-1)} / \prod_{j>k} (a_{ij}^u - a_{ik}^u).$$

We prove this by noting that

$$\sigma_{ju} = \sum_{\Sigma v=r} \prod_{j=1}^i \alpha_{ij}^{uvj} = (-1)^{r(u+1)} \sigma_{iur}$$

and defining

$$h'_r = \sum_{\Sigma v=r} \prod_{j=1}^i \alpha_{ij}^{vj}$$

and showing that

$$h'_r = \sum_{j=1}^i \alpha_{ij}^{r-1} / \prod_{j>k} (\alpha_{ij} - \alpha_{ik}) .$$

It follows from Macmahon [5, p. 4] that h'_r satisfies a linear recurrence relation of order i given by

$$\begin{aligned} h'_r &= \sum_{n=1}^i P_{in} h'_{r-n}, & r > 0, \\ h'_r &= 1, & r = 0 \\ h'_r &= 0, & r < 0; \end{aligned}$$

the P_{ir} and α_{ir} are those of (2.1). We again assume that the α_{ir} are distinct so that from Jarden [4, p. 107]

$$(2.10) \quad h'_r = \sum_{j=1}^i \alpha_{ij}^r D_j / D ,$$

where D is the Vandermonde of the roots, given by

$$(2.11) \quad D = \sum_{j=1}^i \alpha_{ij}^{i-1} \prod_{\substack{j \neq n \neq m \\ n < m}} (\alpha_{im} - \alpha_{in}) = \prod_{j>n} (\alpha_{ij} - \alpha_{in}) \prod_{\substack{j \neq n \neq m \\ n < m}} (\alpha_{im} - \alpha_{in})$$

and D_j is the determinant of order i obtained from D on replacing its j^{th} column by the initial terms of the sequence, $h'_0, h'_1, \dots, h'_{i-1}$. It thus remains to prove that

$$(2.12) \quad D_j = \alpha_{ij}^{i-1} \prod_{\substack{j \neq n \neq m \\ n < m}} (\alpha_{im} - \alpha_{in}) = D \alpha_{ij}^{i-1} / \prod_{j>n} (\alpha_{ij} - \alpha_{in}) .$$

We use the method of the contrapositive. If

$$D_j \neq \alpha_{ij}^{i-1} \prod_{\substack{j \neq n \neq m \\ m > n}} (\alpha_{im} - \alpha_{in}) ,$$

then

$$D = \sum_{j=1}^i D_j$$

(from (2.10) with $n = 0$)

$$\neq \sum_{j=1}^i \alpha_{ij}^{i-1} \prod_{\substack{j \neq n \neq m \\ m > n}} (\alpha_{im} - \alpha_{in})$$

which contradicts (2.11). This proves (2.12) and we have established that

$$h'_r = \sum_{j=1}^i \alpha_{ij}^r D_j / D = \sum_{j=1}^i \alpha_{ij}^{i+n-1} D_j / D \alpha_{ij}^{i-1} = \sum_{j=1}^i \alpha_{ij}^{i+r-1} / \prod_{j>n} (\alpha_{ij} - \alpha_{in}),$$

as required.

3. GENERATING FUNCTION FOR SEQUENCE OF POWERS

Van der Poorten [6] further proved that if

$$(3.1) \quad w^{(i)}(x) = \sum_{n=0}^{\infty} w_n^{(i)} x^n = f(x)/x^i g(x^{-1}),$$

then there exists a polynomial $f_r(x)$ of degree at most $u - 1$, such that

$$(3.2) \quad w_r^{(i)}(x) = f_r(x)/x^u g_r(x^{-1}), \quad u = \binom{r+i-1}{r}.$$

We first seek an expression for $f_r(x)$.

$$\begin{aligned} w_r^{(i)}(x) &= w_0^{(i)r} + w_1^{(i)r} x + w_2^{(i)r} x^2 + \dots + w_{u-1}^{(i)r} x^{u-1} + w_u^{(i)r} x^u + \dots \\ -R_{u1} x w_r^{(i)}(x) &= -R_{u1} w_0^{(i)r} x - R_{u1} w_1^{(i)r} x^2 - \dots - R_{u1} w_{n-2}^{(i)r} x^{u-1} - R_{u1} w_{u-1}^{(i)r} x^u - \dots \\ -R_{u2} x^2 w_r^{(i)}(x) &= -R_{u2} w_0^{(i)r} x^2 - \dots - R_{u2} w_{n-3}^{(i)r} x^{u-1} - R_{u2} w_{u-1}^{(i)r} x^u - \dots \\ &\vdots \\ -R_{u,u-1} x^{u-1} w_r^{(i)}(x) &= -R_{u,u-1} w_0^{(i)r} x^{u-1} - R_{u,u-1} w_1^{(i)r} x^u - \dots \\ -R_{uu} x^u w_r^{(i)}(x) &= -R_{uu} w_0^{(i)r} x^u - \dots \end{aligned}$$

We then sum both sides of these equations. On the left we have

$$w_r^{(i)}(x) \left(1 - \sum_{j=1}^u R_{uj} x^j \right) = w_r^{(i)}(x) x^u \left(x^{-u} - \sum_{j=1}^u R_{uj} x^{-(u-j)} \right) = w_r^{(i)}(x) x^u g_r(x^{-1}),$$

as in van der Poorten.

On the right we obtain

$$(3.3) \quad f_r(x) = \sum_{j=0}^{u-1} T_{uj} x^j,$$

where

$$T_{uj} = w_j^{(i)r} - \sum_{m=0}^j R_{um} w_{j-m}^{(i)r}, \quad R_{u0} \equiv 0,$$

since

$$w_n^{(i)r} x^n = \sum_{j=1}^u R_{uj} w_{n-j}^{(i)r} x^n.$$

Thus we have

$$(3.4) \quad w_r^{(i)}(x) = \left(\sum_{j=0}^{u-1} \left\{ w_j^{(i)r} - \sum_{m=1}^j R_{um} w_{j-m}^{(i)r} \right\} x^j \right) / x^u g_r(x^{-1}).$$

We now show how (3.4) agrees with Eq. (33) of Horadam [3] when $i=2$ and $r=2$. We first multiply each side of the equation by $x^3 g_2(x^{-1})$.

The left-hand side of (3.4) is then

$$\begin{aligned} x^3 g_2(x^{-1}) w_2^{(2)}(x) &= (-1(P_{21}^2 + P_{22})x - (P_{22}^2 + P_{21}^2 P_{22})x^2 + P_{22}^3 x^3) w_2^{(2)}(x) \\ &= (1 + P_{22} x)(1 - (P_{21}^2 + 2P_{22})x + P_{22}^2 x^2) w_2^{(2)}(x). \end{aligned}$$

When $i=2$, the right-hand side of (3.4) is

$$\begin{aligned}
\sum_{j=0}^2 \left\{ w_j^{(2)^2} - \sum_{m=1}^j R_{3m} w_{j-m}^{(2)^2} \right\} x^j &= w_0^{(2)^2} + w_1^{(2)^2} x + w_2^{(2)^2} x^2 - R_{31} w_0^{(2)} x^2 - R_{31} w_1^{(2)} x^2 - R_{32} w_0^{(2)^2} x^2 \\
&= w_0^{(2)^2} + w_1^{(2)^2} x + P_{21} w_1^{(2)^2} x^2 + P_{22}^2 w_0^{(2)^2} x^2 + 2P_{21} P_{22} w_0^{(2)} w_1^{(2)} x^2 \\
&\quad - P_{21} w_0^{(2)^2} x - P_{22} w_0^{(2)^2} x - P_{21} w_1^{(2)^2} x^2 - P_{22} w_1^{(2)^2} x^2 \\
&\quad - P_{22}^2 w_0^{(2)^2} x^2 - P_{21}^2 P_{22} w_0^{(2)^2} x^2 \\
&= (1 + P_{22}x) w_0^{(2)^2} - (1 + P_{22}x)(P_{21} w_0^{(2)} - w_1^{(2)})^2 x \\
&\quad - 2x(P_{21} w_0^{(2)} w_1^{(2)} + P_{22} w_0^{(2)^2} - w_1^{(2)^2}) \frac{(1 + P_{22}x)}{(1 + P_{22}x)} \\
&= (1 + P_{22}x)(w_0^{(2)^2} - x(P_{21} w_0^{(2)} - w_1^{(2)})^2 - 2x w_0^{(2)} (-P_{22}x)) \\
&= (1 + P_{22}x)^{-1}.
\end{aligned}$$

(since $w_0(-P_{22}x)$)

This agrees with Horadam's Eq. (33) if we multiply that equation through by $(1 + P_{22}x)$ and note that $a_{21}^2 + a_{22} = P_{21}^2 + 2P_{22}$. When $r = 1$, we get $u = i$, $R_{im} = P_{im}$. If we consider the special case of $\{w_n^{(i)}\}$:

$$\begin{aligned}
w_n^{(i)} &= 0, & n < 0 \\
w_n^{(i)} &= 1, & n = 0 \\
w_n^{(i)} &= \sum_{r=1}^i P_{ir} w_{n-r}^{(i)}, & n > 0,
\end{aligned}$$

then $\{w_n^{(i)}\} \equiv \{u_n^{(i)}\}$, the fundamental sequence discussed by Shannon [7], and (3.4) becomes

$$\begin{aligned}
u^{(i)}(x) &= \left\{ \sum_{j=0}^{n-1} \left\{ u_j^{(i)} - \sum_{m=1}^j P_{im} u_{j-m}^{(i)} \right\} x^j \right\} / x^i g(x^{-1}) = \left\{ u_0^{(i)} + \sum_{j=1}^{n-1} (u_j^{(i)} - u_j^{(i)}) x^j \right\} / x^i g(x^{-1}) \\
&= 1/x^i g(x^{-1}), \quad \text{where} \quad n = \binom{i+r-1}{r},
\end{aligned}$$

which is effectively Eq. (1) of Hoggatt and Lind [2]. (Equation (2) of Hoggatt and Lind [2] is essentially the same as Eq. (2.4) of Shannon.)

Thus in (2.9) we have found the coefficients in the recurrence relation for $\{w_n^{(i)r}\}$ and in (3.4) an explicit expression for the generating function for $\{w_n^{(i)r}\}$.

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