

SOME ASPECTS OF GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

In a series of papers, Horadam [8], [9], [10], [11] has obtained many results for the generalized Fibonacci sequence $\{H_n\}$ defined below, which he extended to the more general sequence $\{W_n(a,b;p,q)\}$ in [12], [13].

Additional results for the sequence $\{H_n\}$, which we concentrate on here, have been obtained by, among other authors, Iyer [14], and Zeitlin [20]. Some of the results in §5 have been obtained independently by Iyer [14].

It is the purpose of this paper to add to the literature of properties and identities relating to $\{H_n\}$ in the expectation that they may prove useful to Fibonacci researchers. Further material relating to properties of $\{H_n\}$ will follow in another article.

Though these results may be exhausting to the readers, they are not clearly exhaustive of the rich resources opened up. As Descartes said in another context, we do not give all the facts but leave some so that their discovery may add to the pleasure of the reader.

2. A GENERATION OF H_n

Generalized Fibonacci numbers H_n are defined by the second-order recurrence relation

$$(2.1) \quad H_{n+2} = H_{n+1} + H_n \quad (n \geq 0)$$

with initial conditions

$$(2.2) \quad H_0 = q, \quad H_1 = p$$

and the proviso that H_n may be defined for $n < 0$.

(See Horadam [12].)

Standard methods (*e.g.* use of difference equations), allow us to discover that

$$(2.3) \quad H_n = \frac{1}{2\sqrt{5}} \left(\varrho a^n - m \beta^n \right)$$

where

$$(2.4) \quad \left\{ \begin{array}{l} \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} \quad (\text{roots of } x^2 - x - 1 = 0), \text{ so that} \\ \alpha + \beta = 1, \quad \alpha\beta = -1, \quad \alpha - \beta = \sqrt{5}, \quad \beta = -\alpha^{-1}; \\ \varrho = 2(p - q\beta), \quad m = 2(p - q\alpha), \quad \text{so that} \\ \varrho + m = 2(2p - q), \quad \varrho - m = 2q\sqrt{5} \quad \text{and} \\ \frac{1}{2}\varrho m = p^2 - pq - q^2 = d \quad (\text{say}). \end{array} \right.$$

It is well known that $p = 1, q = 0$ leads to the ordinary Fibonacci sequence $\{F_n\}$, while $p = 2q = -1$ leads to the Lucas sequence $\{L_n\}$.

Following an analytic procedure due to Hagis [5] for generating the ordinary Fibonacci number F_n , we proceed to an alternative establishment of (2.3).

Put $h_n = H_{n+1}$. Let

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Letting C_j^n ($j = 0, 1, 2, \dots, n, \dots$) be an element of this array, where the superscript refers to rows and the subscripts to columns, we define the array as in Gould [3] by the conditions:

$$(3.1) \quad C_0^0 = C_0^1 = p, \quad C_1^1 = q$$

$$(3.2) \quad C_j^n = 0 \quad \text{if } j > n \text{ or } j < 0 \quad .$$

$$(3.3) \quad C_j^{n+1} = C_{j-1}^n + \frac{1+(-1)^j}{2} C_j^n \quad \text{if } n \geq 1, j \geq 0.$$

The row-sums are given by

$$(3.4) \quad S_n(p, q) = \sum_{j=0}^n C_j^n \quad (n \geq 0) \\ = pF_{n+1} + qF_n = H_{n+1}$$

by Horadam [8]. Thus the row-sums of this array generate the generalized Fibonacci numbers. As indicated in Gould [3] the given array generalizes two variants of Pascal's triangle which are related to Fibonacci numbers and to Lucas numbers.

It may easily be verified that

$$(3.5) \quad C_{2k}^n = \binom{n-k-1}{k} p + \binom{n-k-1}{k-1} q$$

$$(3.6) \quad C_{2k+1}^n = \binom{n-k-2}{k} p + \binom{n-k-2}{k-1} q$$

so that

$$(3.7) \quad \sum_{j=0}^n C_j^n = \sum_{k=0}^{[n/2]} C_{2k}^n + \sum_{k=0}^{[(n-1)/2]} C_{2k+1}^n \\ = H_{n+1} \quad ,$$

as expected (cf. (3.4)).

Similarly, we can show that

$$(3.8) \quad \sum_{j=0}^n (-1)^j C_j^n = H_{n-2}, \quad n \geq 2.$$

If we define the polynomials $\{C_n(x)\}$ by

$$(3.9) \quad C_n(x) = \sum_{j=0}^n C_j^n x^j \quad ,$$

then we have on using (3.5) and (3.6) that

$$(3.10) \quad C_n(x) = \sum_{k=0}^{[n/2]} \left\{ \binom{n-k-1}{k} p + \binom{n-k-1}{k-1} q \right\} x^{2k} \\ + \sum_{k=0}^{[(n-1)/2]} \left\{ \binom{n-k-2}{k} p + \binom{n-k-2}{k-1} q \right\} x^{2k+1} \quad ,$$

where it can be shown, as in Gould [3], that the polynomial $C_n(x)$ satisfies the simple recurrence relation

$$(3.11) \quad 2C_{n+1}(x) = (2x+1)C_n(x) + C_n(-x)$$

on using (3.3). Similarly, it can be shown that $C_n(x)$ satisfies the second-order recurrence relation

$$(3.12) \quad C_{n+2}(x) = C_{n+1}(x) + x^2 C_n(x) \quad .$$

It may be noted in passing that certain properties of an array involving the elements of $\{H_n\}$ are given in Wall [19].

4. GENERALIZED FIBONACCI FUNCTIONS

Elmore [1] described the concept of Fibonacci functions. Extending his idea, we have a sequence of generalized Fibonacci functions $\{H_n(x)\}$ if we put

$$(4.1) \quad \left\{ \begin{aligned} H_0(x) &= \frac{1}{2\sqrt{5}} \{ \alpha e^{\alpha x} - m e^{\beta x} \} \\ H_1(x) &= H'_0(x) \\ H_2(x) &= H''_0(x) \\ &\dots\dots\dots \\ H_n(x) &= H^{(n)}_0(x) = \frac{1}{2\sqrt{5}} \{ \alpha^n e^{\alpha x} - m \beta^n e^{\beta x} \} \end{aligned} \right.$$

so that we have

$$(4.2) \quad H_{n+2}(x) = H_{n+1}(x) + H_n(x) .$$

Obviously,

$$(4.3) \quad \begin{aligned} H_0(0) &= q = H_0, \quad H_1(0) = p = H_1, \\ H_2(0) &= p + q = H_2, \dots, \end{aligned}$$

etc., and

$$(4.4) \quad H_n(0) = \frac{1}{2\sqrt{5}} \{ \alpha^n - m \beta^n \} = H_n .$$

We are able to find numerous identities for these generalized Fibonacci functions, some of which are listed below for reference:

$$(4.5) \quad H_{n-1}(x)H_{n+1}(x) - H_n^2(x) = (-1)^n de^x$$

$$(4.6) \quad H_{n-1}(x)F_r(x) + H_n(x)F_{r+1}(x) = H_{n+r}(2x) ,$$

where the $F_n(x)$ are the Fibonacci functions corresponding to the $f_n(x)$ of Elmore [1]. Similarly,

$$(4.7) \quad H_{n-1}(u)F_r(v) + H_n(u)F_{r+1}(v) = H_{n+r}(u+v)$$

$$(4.8) \quad H_{n-1}^2(x) + H_n^2(x) = (2p - q)H_{2n-1}(2x) - dF_{2n-1}(2x)$$

$$(4.9) \quad H_{n+1}^2(x) - H_n^2(x) = (2p - q)H_{2n}(2x) - dF_{2n}(2x)$$

$$(4.10) \quad H_n^3(x) + H_{n+1}^3(x) = 2H_n(x)H_{n+1}^2(x) + (-1)^n de^x H_{n-1}(x)$$

$$(4.11) \quad H_{n+1-r}(x)H_{n+1+r}(x) - H_{n+1}^2(x) = (-1)^{n-r} de^x F_r^2$$

$$(4.12) \quad H_n(x)H_{n+1+r}(x) - H_{n-s}(x)H_{n+r+s+1}(x) = (-1)^{n-s} de^x F_s F_{r+s+1} .$$

We note here that (8) to (14) of Horadam [8] are a special case of (4.5) to (4.12) above, since, as we have already shown in (4.3) and (4.4), the generalized Fibonacci functions become the generalized Fibonacci numbers $\{H_n\}$ when $x = 0$.

As in Horadam [8], we also note that (4.5) is a special case of (4.11) when $r = 1$ and n is replaced by $n - 1$. If we put $r = n$ in (4.11) we have

$$(4.13) \quad H_1(x)H_{2n+1}(x) - H_{n+1}^2(x) = de^x F_n^2 .$$

Corresponding to the Pythagorean results in Horadam [8], we have, for the generalized Fibonacci function $H_n(x)$

$$(4.14) \quad \{2H_{n+1}(x)H_{n+2}(x)\}^2 + \{H_n(x)H_{n+3}(x)\}^2 = \{2H_{n+1}(x)H_{n+2}(x) + H_n^2(x)\}^2$$

from which we may derive (16) of Horadam [8], for the special case when $x = 0$.

The above identities are easily established by use of the formula for $H_n(x)$ given in (4.1) with reference to the identities

$$(4.15) \quad \left\{ \begin{array}{l} 1 + \alpha^2 = \alpha\sqrt{5} \quad , \quad 1 + \beta^2 = -\beta\sqrt{5} \quad , \\ \alpha\beta = -1 \quad , \quad \frac{1}{2} \alpha m = d \quad , \\ \alpha^3 = 2 + \sqrt{5} \quad , \quad 1 + \alpha^3 = 2\alpha^2 \quad , \\ 2\alpha + \beta = \alpha^2 \quad , \quad 1 + \alpha = \alpha^2 \quad , \\ \alpha + \beta = 1 \quad , \quad \alpha(2\rho - q) - 2d = \frac{1}{2}\alpha^2 \quad , \text{ etc.} \end{array} \right.$$

As in Elmore [1], we can extend this theory of generalized Fibonacci functions to generalized Fibonacci functions of two variables to give a function of two variables, thus:

$$(4.16) \quad \phi_0 \equiv \phi(x, y) = \sum_{i=0}^{\infty} H_i(x) \frac{y^i}{i!} = H_0(x) + H_1(x)y + H_2(x) \frac{y^2}{2!} + \dots$$

Differentiating (4.16) term-by-term gives

$$(4.17) \quad \frac{\partial \phi_0}{\partial x} = \sum_{i=1}^{\infty} H_i(x) \frac{y^{i-1}}{(i-1)!} = \sum_{i=0}^{\infty} H_{i+1}(x) \frac{y^i}{i!}$$

$$\frac{\partial \phi_0}{\partial y} = \sum_{i=0}^{\infty} H_{i+1}(x) \frac{y^i}{i!}$$

i.e.,

$$(4.18) \quad \frac{\partial \phi_0}{\partial x} = \frac{\partial \phi_0}{\partial y} .$$

Similarly, we can verify that all higher power partial derivatives are equal, so that if we denote the k th partial derivative by ϕ_k , we have

$$(4.19) \quad \phi_k = \frac{\partial^k \phi}{\partial x^r \partial y^s} = \sum_{i=0}^{\infty} H_{k+i}(x) \frac{y^i}{i!} = \sum_{i=0}^{\infty} H_{k+i}(y) \frac{x^i}{i!} ,$$

where r and s are positive integers such that $r + s = k$. Noting that

$$(4.20) \quad \phi_k(x, 0) = H_k(x), \quad \phi_k(0, y) = H_k(y), \quad \phi_k(0, 0) = H_k ,$$

we can expand $\phi_k(x, y)$ as a power series of the two variables x and y at $(0, 0)$ so that we have

$$(4.21) \quad \begin{aligned} \phi_k(x, y) &= \phi_k(0, 0) + \left[x \frac{\phi_k(0, 0)}{\partial x} + y \frac{\phi_k(0, 0)}{\partial y} \right] \\ &+ \frac{1}{2!} \left[x^2 \frac{\partial^2 \phi_k(0, 0)}{\partial x^2} + 2xy \frac{\partial^2 \phi_k(0, 0)}{\partial x \partial y} + y^2 \frac{\partial^2 \phi_k(0, 0)}{\partial y^2} \right] + \dots \\ &= H_k + H_{k+1} \frac{(x+y)}{1!} + H_{k+2} \frac{(x+y)^2}{2!} + \dots \end{aligned}$$

so that

$$(4.22) \quad \phi_k(x, y) = H_k(x+y) = \frac{\alpha \alpha^k e^{\alpha(x+y)} - m \beta^k e^{\beta(x+y)}}{2(\alpha - \beta)} .$$

5. GENERALIZED FIBONACCI NUMBER IDENTITIES

Many other interesting and useful identities may be derived for the sequence $\{H_n\}$ using inductive methods or by argument from (2.1). We list some elementary results without proof:

$$(5.1) \quad H_{-n} = (-1)^n [qF_{n+1} - pF_n]$$

$$(5.2) \quad \sum_{i=0}^n H_i = H_{n+2} - H_1 [= H_{n+2} - p]$$

$$(5.3) \quad \sum_{i=0}^n H_{2i-1} = H_{2n} - H_{-2} [= H_{2n} + (p - 2q)]$$

$$(5.4) \quad \sum_{i=0}^n H_{2i} = H_{2n+1} - H_{-1} [= H_{2n+1} - (p - q)]$$

$$(5.5) \quad \sum_{i=0}^{2n} (-1)^{i+1} H_i = -H_{2n-1} + p - 2q$$

$$(5.6) \quad \sum_{i=0}^n H_i^2 = H_n H_{n+1} - q(p - q)$$

$$(5.7) \quad \sum_{i=0}^n i H_i = (n + 1) H_{n+2} - H_{n+4} + H_3$$

$$(5.8) \quad \sum_{i=0}^n \binom{n}{i} H_i = H_{2n}$$

$$(5.9) \quad \sum_{i=0}^n \binom{n}{i} H_{3i} = 2^n H_{2n}$$

$$(5.10) \quad \sum_{i=0}^n \binom{n}{i} H_{4i} = 3^n H_{2n} .$$

The three summations (5.8), (5.9) and (5.10), which are generalizations of similar results for the ordinary Fibonacci Sequence $\{F_n\}$ as listed in Hoggatt [6], may all be proved by numerical substitution as, for example, in

$$\begin{aligned} \sum_{i=0}^n \binom{n}{i} H_{3i} &= \frac{1}{2\sqrt{5}} \left\{ \sum_{i=0}^n \binom{n}{i} \alpha^{3i} - m \sum_{i=0}^n \binom{n}{i} \beta^{3i} \right\} \\ &= \frac{1}{2\sqrt{5}} \left\{ \alpha (1 + \alpha^3)^n - m (1 + \beta^3)^n \right\} \\ &= \frac{2^n}{2\sqrt{5}} \left\{ \alpha \alpha^{2n} - m \beta^{2n} \right\} = 2^n H_{2n} . \end{aligned}$$

Some further generalizations of identities listed in Subba Rao [17] are:

$$(5.11) \quad \sum_{i=0}^n H_{3i-2} = \frac{1}{2} (H_{3n} - H_{-3})$$

Proof: Using identity (3) of Horadam [8], viz.,

$$2H_n = H_{n+2} - H_{n-1} .$$

we have

$$2H_{-2} = H_0 - H_{-3}$$

$$2H_{-1} = H_3 - H_0$$

.....

Adding both sides and then dividing by two gives the desired result. Similarly,

$$(5.12) \quad \sum_{i=0}^n H_{3i-1} = \frac{1}{2}(H_{3n+1} - H_{-2})$$

$$(5.13) \quad \sum_{i=0}^n H_{3i} = \frac{1}{2}(H_{3n+2} - H_{-1}) .$$

Some additional identities corresponding to formulae for the sequence $\{F_n\}$ in Siler [16], are

$$(5.14) \quad \sum_{i=0}^n H_{4i-3} = F_{2(n+1)}H_{2n-3}$$

$$(5.15) \quad \sum_{i=0}^n H_{4i-1} = F_{2(n+1)}H_{2n-1}$$

$$(5.16) \quad \sum_{i=0}^n H_{4i-2} = F_{2(n+1)}H_{2n-2}$$

$$(5.17) \quad \sum_{i=0}^n H_{4i} = F_{2(n+1)}H_{2n} .$$

As in Siler [16], identities (5.4) and (5.11) to (5.17) suggest that we should be able to solve the general summation formula

$$(5.18) \quad \sum_{i=1}^n H_{ai-b} .$$

Proceeding as in Siler [16], we have:

$$\begin{aligned} \sum_{i=1}^n H_{ai-b} &= \frac{1}{2\sqrt{5}} \left\{ \alpha \sum_{i=1}^n \alpha^{ai-b} - m \sum_{i=1}^n \beta^{ai-b} \right\} \\ &= \frac{(-1)^a H_{an-b} - H_{a(n+1)-b} - (-1)^a H_{-b} + H_{a-b}}{(-1)^a + 1 - L_a} \end{aligned}$$

on using the fact that

$$\sum_{i=1}^n \alpha^{ai-b} = \alpha^{a-b} \underbrace{[1 + \alpha^a + \dots + \alpha^{(n-1)a}]_{n \text{ terms}}} = \alpha^{a-b} \frac{\alpha^{na} - 1}{\alpha^a - 1}$$

with a similar expression for the term involving β . Here it should be stated that Siler rediscovered a special case due to Lucas in 1878.

The identity (5.19) below which arose as a generalization of the combination of (2) and (3) of Sharpe [15], may be established thus:

$$(5.19) \quad H_{n+2k+1}^2 + H_{n+2k}^2 = H_{2k+1}H_{2n+2k+1} + H_{2k}H_{2n+2k}$$

Proof:

$$\begin{aligned} 20(H_{n+2k+1}^2 + H_{n+2k}^2) &= (\varrho\alpha^{n+2k+1} - m\beta^{n+2k+1})^2 + (\varrho\alpha^{n+2k} - m\beta^{n+2k})^2 \\ &= \varrho^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \varrho^2\alpha^{2n+4k} + m^2\beta^{2n+4k} - 8d(\alpha\beta)^{n+2k} [1 + \alpha\beta] \\ &= \varrho^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \varrho^2\alpha^{2n+4k} + m^2\beta^{2n+4k} \\ 20(H_{2k+1}H_{2n+2k+1} + H_{2k}H_{2n+2k}) &= \varrho^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \varrho^2\alpha^{2n+4k} + m^2\beta^{2n+4k} \\ &\quad - \varrho m(\alpha\beta)^{2k+1} [\alpha^{2n} + \beta^{2n}] - \varrho m(\alpha\beta)^{2k} [\alpha^{2n} + \beta^{2n}] \\ &= \varrho^2\alpha^{2n+4k+2} + m^2\beta^{2n+4k+2} + \varrho^2\alpha^{2n+4k} + m^2\beta^{2n+4k} \end{aligned}$$

In an attempt to generalize those identities found in Tadlock [18], involving the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ we have

$$(5.20) \quad F_{2j+1} \mid (H_{k+j+1}^2 + H_{k-j}^2)$$

Proof:

$$\begin{aligned} H_{k+j+1}^2 + H_{k-j}^2 &= \left[\frac{\varrho\alpha^{k+j+1} - m\beta^{k+j+1}}{2(\alpha - \beta)} \right]^2 + \left[\frac{\varrho\alpha^{k-j} - m\beta^{k-j}}{2(\alpha - \beta)} \right]^2 \\ &= \frac{\varrho^2\alpha^{2k+1}(\alpha^{2j+1} + \alpha^{-2j-1}) + m^2\beta^{2k+1}(\beta^{2j+1} + \beta^{-2j-1})}{4(\alpha - \beta)^2} \\ &\quad - \frac{2d(\alpha\beta)^{k+j}[\alpha\beta + (\alpha\beta)^{-2j}]}{(\alpha - \beta)^2} \\ &= \frac{(\alpha^{2j+1} - \beta^{2j+1})(\varrho^2\alpha^{2k+1} - m^2\beta^{2k+1})}{(\alpha - \beta)4(\alpha - \beta)} \end{aligned}$$

since

$$\begin{cases} \alpha^{-2j-1} = -\beta^{2j+1} \\ \beta^{-2j-1} = -\alpha^{2j+1} \end{cases}$$

i.e.,

$$H_{k+j+1}^2 + H_{k-j}^2 = F_{2j+1} \cdot \frac{\varrho^2\alpha^{2k+1} - m^2\beta^{2k+1}}{\alpha - \beta}$$

i.e.,

$$F_{2j+1} \mid (H_{k+j+1}^2 + H_{k-j}^2).$$

Also,

$$(5.21) \quad 2[2H_n^2 + (-1)^n d]^2 = H_n^4 + H_{n+1}^4 + H_{n-1}^4.$$

This identity which is a generalization of Problem H-79 proposed by Hunter [7], may be solved as follows. From the identity (11) of Horadam [8], we have

$$(5.22) \quad \begin{aligned} 2[2H_n^2 + (-1)^n d] &= 2[H_{n-1}H_{n+1} + H_n^2]^2 \\ &= H_n^4 + H_n^4 + 4H_n^2H_{n-1}H_{n+1} + 2H_{n-1}^2H_{n+1}^2. \end{aligned}$$

Now,

$$(5.23) \quad H_n^4 + 4H_n^2H_{n-1}H_{n+1} + 2H_{n-1}^2H_{n+1}^2 = (H_{n+1} - H_{n-1})^4 + 4(H_{n+1} - H_{n-1})^2H_{n-1}H_{n+1}$$

on calculation, so that (5.21) follows from (5.22) and (5.23).

Two further interesting results are obtained by considering the following generalization of Problem B-9 proposed by Graham [4]. From

$$\frac{1}{H_{n-1}H_{n+1}} = \frac{H_n}{H_{n-1}H_nH_{n+1}} = \frac{H_{n+1} - H_{n-1}}{H_{n-1}H_nH_{n+1}} = \frac{1}{H_{n-1}H_n} - \frac{1}{H_nH_{n+1}}$$

we have, on summing both sides over $n = 2, \dots, \infty$,

$$(5.24) \quad \sum_{n=2}^{\infty} \frac{1}{H_{n-1}H_{n+1}} = \frac{1}{p(p+q)} .$$

Similarly, from

$$\frac{H_n}{H_{n-1}H_{n+1}} = \frac{H_{n+1} - H_{n-1}}{H_{n-1}H_{n+1}} = \frac{1}{H_{n-1}} - \frac{1}{H_{n+1}}$$

we have

$$(5.25) \quad \sum_{n=2}^{\infty} \frac{H_n}{H_{n-1}H_{n+1}} = \frac{2p+q}{p(p+q)} .$$

6. RECURRENCE RELATIONS FOR $\{H_n\}$

If we define a sequence $\{G_n\}$ by $G_n = H_{H_n}$, and define $\{X_n\}$ and $\{Y_n\}$ by $X_n = F_{H_n}$ and $Y_n = L_{H_n}$, then we may verify that

$$(6.1) \quad G_{n+3} = G_{n+2}Y_{n+1} - (-1)^{H_{n+1}}G_n ,$$

which corresponds exactly with (1) of Ford [2], and that

$$(6.2) \quad 2G_{n+3} = G_{n+1}Y_{n+2} + G_{n+2}Y_{n+1} - (-1)^{H_{n+1}}H_0Y_n$$

corresponding to (5) of Ford [2].

If we now define the sequence $\{Z_n\}$ by $Z_n = H_{H_{n+j}}$, then

$$(6.3) \quad \begin{aligned} Z_n &= \frac{1}{2\sqrt{5}} \left\{ \alpha^j R_n - m\beta^j S_n \right\} \\ &= \frac{1}{2\sqrt{5}} \left\{ \alpha^j R_n - m\beta^j S_n \right\} , \end{aligned}$$

where $R_n = \alpha^{H_n}$ (and $S_n = \beta^{H_n}$) for convenience.

$$(6.4) \quad \begin{aligned} \therefore Z_{n+2} &= \frac{1}{2\sqrt{5}} \left\{ \alpha^j R_{n+2} - m\beta^j S_{n+2} \right\} \\ &= \frac{1}{2\sqrt{5}} \left\{ \alpha^j R_{n+1}R_n - m\beta^j S_{n+1}S_n \right\} \end{aligned}$$

since $R_{n+2} = \alpha^{H_{n+2}} = \alpha^{H_{n+1}}\alpha^{H_n} = R_{n+1}R_n$, and similarly for S_{n+2} .

$$(6.5) \quad \begin{aligned} \therefore Z_{n+2} &= \frac{1}{2\sqrt{5}} \left\{ R_n(\alpha^j R_{n+1} - m\beta^j S_{n+1}) + S_{n+1}(\alpha^j R_n - m\beta^j S_n) \right. \\ &\quad \left. - R_n S_{n+1}(\alpha^j - m\beta^j) \right\} = R_n Z_{n+1} + S_{n+1} Z_n - R_n S_{n+1} H_j \end{aligned}$$

i.e.,

$$(6.6) \quad Z_{n+2} = R_n Z_{n+1} + S_{n+1} Z_n - (-1)^{H_n} S_{n-1} H_j$$

since

$$R_n S_{n+1} = \alpha^{H_n} \beta^{H_{n+1}} = (\alpha\beta)^{H_n} \beta^{H_{n-1}} .$$

Similarly,

$$(6.7) \quad Z_{n+2} = S_n Z_{n+1} + R_{n+1} Z_n - (-1)^{H_n} R_{n-1} H_j .$$

Adding Eqs. (6.6) and (6.7) gives

$$(6.8) \quad 2Z_{n+2} = Z_{n+1}(R_n + S_n) + Z_n(R_{n+1} + S_{n+1}) - (-1)^{H_n} H_j (R_{n-1} + S_{n-1})$$

i.e.,

$$2Z_{n+2} = Y_n Z_{n+1} + Y_{n+1} Z_n - (-1)^{H_n} Y_{n-1} H_j$$

since

$$R_n + S_n = \alpha^{H_n} + \beta^{H_n} = L_{H_n} = Y_n$$

i.e.

$$(6.9) \quad 2H_{H_{n+2+j}} = L_{H_n} H_{H_{n+1}} + L_{H_{n+1}} H_{H_{n+j}} - (-1)^{H_n} L_{H_{n-1}} H_j$$

which is a generalization of (14) of Ford [2].

One can continue discovering new generalizations *ad infinitum* (but not, we hope, *ad nauseam!*), but the time has come for a halt.

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