# A LOWER BOUND FOR THE PERIOD OF THE FIBONACCI SERIES MODULO $M$

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In this note we shall determine a nontrivial lower bound for the period of the Fibonacci series modulo $m$. This problem was posed by D. D. Wall [2], p. 529.

Let $a(m)$ denote the subscript of the first term of the Fibonacci series

(1) $$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \qquad F_1 = 1,$$

which is divisible by $m$. Let $k(m)$ denote the period of $\{F_n\}$ modulo $m$. Define the sequence $\{L_n\}$ so that

(2) $$L_{n+2} = L_{n+1} + L_n, \qquad L_0 = 2, \qquad L_1 = 1.$$

Then our main result is the following theorem:

***Theorem.*** Let $t$ be any natural number such that $L_t \leqslant m$, where $m > 2$. Then $k(m) \geqslant 2t$, with equality if and only if $L_t = m$ and $t$ is odd.

Wall posed the question for prime values of $m$. It is not known whether or not there are infinitely many prime $m$ such that $L_t = m$ when $t$ is odd.

For the proof of the theorem we need some preliminary results. The following theorem is proved in [1] (Th. 3).

***Vinson's Theorem.*** Let $m$ be any integer greater than 2. If $a(m)$ is odd, then $k(m) = 4a(m)$; if $8 \nmid m$ and $a(p) \equiv 2 \pmod 4$ for all odd prime divisors of $m$, then $k(m) = a(m)$; in any other case, $k(m) = 2a(m)$.

In addition to well known identities, the following is useful:

(3) $$F_i \equiv -(-1)^i F_{k(m)-i} \pmod m.$$

Equation (3) follows by induction on $i$, using (1), $F_0 \equiv F_{k(m)} \equiv 0$, and $F_1 \equiv F_{k(m)+1} \equiv 1 \pmod m$.

***Lemma.*** $k(L_n) = 4n$ when $n$ is even; $k(L_n) = 2n$ when $n$ is odd.

***Proof:*** $a(L_n) = 2n$ is known, and may be proved using $F_{2n} = F_n L_n$, $F_t < L_n$ for $t \leqslant n+1$, and the fact that subscripts $n$ for which $m \mid U_n$ form an ideal. The lemma follows by an application of Vinson's theorem.

***Proof of Theorem:*** It is known [2] that $k(m)$ is even. Using the identities (3) and

(4) $$L_n = F_{n-1} + F_{n+1},$$

we see that if $k(m) = 2t$ then $F_t \equiv -(-1)^t F_t \pmod m$, so $t$ is odd or $F_t \equiv 0 \pmod m$. If $t$ is odd then by (3) we have $F_{t+1} \equiv -F_{t-1} \pmod m$ implying (by (4)) that

(5) $$L_t \equiv F_{t+1} + F_{t-1} \equiv 0 \pmod m.$$

Otherwise, if $F_t \equiv 0 \pmod m$ then by (1),

(6) $$F_{t+1} - F_{t-1} \equiv 0 \pmod m.$$

Clearly, if $t \leqslant n$ then

(7) $$0 < F_{t+1} + F_{t-1} \leqslant F_{n+1} + F_{n-1} = L_n \leqslant m,$$

by the hypothesis of the theorem. By (4)

$$F_{t+1} - F_{t-1} = 2F_{t+1} - L_t,$$

and since $m \geqslant L_t > F_{t+1}$ when $t \leqslant n$, we have

(8) $$F_{t+1} - F_{t-1} < m.$$

Now, (5) and (7) imply that $t = n$ and $L_t = m,$ and (6) and (8) are never simultaneously true. Thus $t \geqslant n,$ with equality only if $L_n = m.$ By the lemma,

$$k(m) = 2t = 2n$$

if and only if $n$ and $t$ are odd and $L_n = m.$ The conclusion of the theorem follows.

## REFERENCES


1. John Vinson, "The Relation of the Period Modulo $m$ to the Rank of Apparition of $m$ in the Fibonacci Sequence," *The Fibonacci Quarterly,* Vol. 1, No. 2 (April, 1963), pp. 37–45.
2. D.D. Wall, "Fibonacci Series Modulo $m$," *Amer. Math. Monthly,* 67 (1960), pp. 525–532.


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$$H_k(x) = \sum_{n=0}^{\infty} H_n^k x^n \qquad (H_k(0) = (H_o)^k = r^k)\ ,$$

where

$$H_o(x) = f_0(x) = \sum_{n=0}^{\infty} x^n = (1-x)^{-1}$$

and

$$H_1(x) = (r+sx)(1-x-x^2)^{-1}$$

are

$$(2)\qquad \begin{cases} (1-3x+x^2)H_2(x) = r^2 - s^2x - 2exH_0(-x) \\ (1-4x-x^2)H_3(x) = r^3 + s^3x - 3exH_1(-x) \\ (1-7x+x^2)H_4(x) = r^4 - s^4x + 2e^2xH_0(x) - 8exH_2(-x) \\ (1-11x-x^2)H_5(x) = r^5 + s^5x + 5e^2xH_1(x) - 15exH_3(-x)\ . \end{cases}$$

The general expression for the generating function is (see [3])

$$(3)\qquad (1-a_kx+(-1)^kx^2)H_k(x) = r^k - (-s)^kx + kx \sum_{j=1}^{[k/2]} \frac{(-1)^j}{j} e^j a_{kj} H_{k-2j}((-1)^j x)\ ,$$

where

$$(1-x-x^2)^{-j} = \sum_{k=2j}^{\infty} a_{kj} x^{k-2j}\ ,$$

that is, $a_{kj}$ are generated by the $j^{th}$ power of the generating function for Fibonacci numbers $f_n$. Note the occurrence in (3) of the Lucas numbers $a_n$.

## FUNCTIONS ASSOCIATED WITH THE GENERATING FUNCTIONS

In the process of obtaining (3), we use

$$(4)\qquad g_k(x) = \sqrt{5}\, H_k(x) = \sum_{j=0}^{[k/2]} \binom{k}{j} e^j F_{k-2j}((-1)^j x) \qquad (F_0(x) = H_0(x))\ ,$$

where

$$F_k(x) = [(r-sb)a]^k(1-a^kx)^{-1} + [(sa-r)b]^k(1-b^kx)^{-1} \qquad (k = 1,2,3,\cdots)$$

and

$$a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2} \qquad (a,b \text{ roots of } x^2-x-1 = 0),$$

leading to the general inverse

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