COMBINATORIAL INTERPRETATION OF AN ANALOG OF GENERALIZED BINOMIAL COEFFICIENTS

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1. INTRODUCTION

Defining $f_{j,k}(n;r,s)$ as the number of sequences of nonnegative integers

$$\left\{a_1, a_2, \cdots, a_n\right\} \tag{1.1}$$

such that

$$-s \leqslant a_{i+1} - a_i \leqslant r \qquad (1 \leqslant i \leqslant n-1)\,, \tag{1.2}$$

where r and s are arbitrary positive integers, and

$$a_1 = j, \qquad a_n = k\,, \tag{1.3}$$

the author [2] has shown that the generating function

$$\phi_{j,r,s}(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\min\left\{n(r+s),j+nr\right\}} f_{j,j+nr-m}(n+1;r,s)x^n y^m$$

can be expressed in terms of generalized binomial coefficients $c_{r+s}(n,k)$ defined by

$$\left(\sum_{h=0}^{r+s} x^h\right)^n = \sum_{k=0}^{\infty} c_{r+s}(n,k)x^k\,. \tag{1.4}$$

For the cases $r = 1$ or $s = 1$ we have explicit formulas for $f_{j,k}(n;r,s)$, namely

$$f_{j,k}(n+1;1,s) = \sum_{t=0}^{j} c_{s+1}(-t-1,j-t)\left[c_{s+1}(n+t,n+t-k) - \sum_{h=0}^{s-1}(h+1)c_{s+1}(n+t,n+t-k-h-2)\right]\,. \tag{1.5}$$

and

$$f_{j,k}(n+1;r,1) = \sum_{t=0}^{k} c_{r+1}(-t-1,k-t)\left[c_{r+1}(n+t,n+t-j) - \sum_{h=0}^{r-1}(h+1)c_{r+1}(n+t,n+t-j-h-2)\right]. \tag{1.6}$$

These formulas generalize a result of Carlitz [1] for $r = s = 1$.

We now define an analog of $c_{r+s}(n,k)$, $n > 0$, by

$$\prod_{j=1}^{n}\left(\sum_{h=0}^{r+s} q^{(r-h)j}x^h\right) = \sum_{k=0}^{n(r+s)} c_{r+s}(n,k;q)x^k\,. \tag{1.7}$$

Letting $f_k(m,n;r,s)$ denote the number of sequences of integers

$$\left\{a_1, a_2, \cdots, a_n\right\} \tag{1.8}$$

satisfying

$$-s \leqslant a_{i+1} - a_i \leqslant r \qquad (1 \leqslant i \leqslant n-1)\,, \tag{1.9}$$

where r and s are nonnegative integers,

$$a_1 = 0, \qquad a_n = k \tag{1.10}$$

and

(1.11)
$$\sum_{i=1}^{n} a_i = m ,$$

we show in this paper that

(1.12)
$$c_{r+s}(n,k;q) = \sum_m f_{nr-k}(m, n+1; r,s)q^m .$$

From (1.12) we obtain a partition identity.

2. COMBINATORIAL INTERPRETATION OF $c_{r+s}(n,k;q)$

From the definition of $f_k(m,n;r,s)$ it follows that

(2.1)
$$f_k(m,1;r,s) = \delta_{k,0}\delta_{m,0}$$

and

(2.2)
$$f_k(m,n+1;r,s) = \sum_{h=0}^{r+s} f_{k+s-h}(m-k, n; r,s) .$$

Now (2.1) had (2.2) imply respectively

(2.3)
$$\sum_m f_k(m,1;r,s)q^m = \delta_{k,0}$$

and

(2.4)
$$\sum_m f_k(m,n+1;r,s)q^m = \sum_{h=0}^{r+s}\sum_m f_{k+s-h}(m,n;r,s)q^{m+k} .$$

Let

$$\phi(x,y;q) = \sum_{n=0}^{\infty}\sum_{k=0}^{n(r+s)}\sum_m f_{nr-k}(m,n+1;r,s)q^m x^k y^n .$$

Using (2.3) and (2.4) we get

$$\phi(x,y;q) = 1 + y\sum_{h=0}^{r+s}\sum_{n=0}^{\infty}\sum_{k=0}^{(n+1)(r+s)}\sum_m f_{nr-k+h}(m,n+1;r,s)q^{m+nr-k}x^k y^r = 1 + y\left(\sum_{h=0}^{r+s} q^{r-h}x^h\right)\phi(xq^{-1},yq^r;q).$$

By iteration

$$\phi(x,y;q) = \sum_{n=0}^{\infty}\prod_{j=1}^{n}\left(\sum_{h=0}^{r+s} q^{(r-h)j}x^h\right)y^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n(r+s)} c_{r+s}(n,k;q)x^k y^n .$$

Equating coefficients we have

(2.5)
$$c_{r+s}(n,k;q) = \sum_m f_{nr-k}(m,n+1;r,s)q^m .$$

3. APPLICATION TO PARTITIONS

Assuming the parts of a partition to be written in ascending order, let $u_r(k,m,n)$ denote the number of partitions of m into at most n parts with the minimum part at most r, the maximum part k and the difference between consecutive parts at most r. Define $v_r(k,m,n)$ to be the number of partitions of m into k parts with each part at most n and each part occurring at most r times. We show that

(3.1)
$$u_r(k,m,n) = v_r(k,m,n) \qquad (r \geq 1).$$

Proof. It is easy to see that

(3.2)
$$u_r(k,m,n) = f_k(m,n+1;r,0) .$$

By (2.5) and (1.7) we have

$$\sum_{k=0}^{nr}\sum_m f_k(m,n+1;r,0)q^m x^k = \sum_{k=0}^{nr} c_r(n,nr-k;q)x^k = \prod_{j=1}^{n}\left(\sum_{h=0}^{r} q^{hj}x^h\right)$$

Thus the generating function for $u_r(k,m,n)$ is

$$\prod_{j=1}^{n} \left(\sum_{h=0}^{r} q^{hj} x^h \right) . \tag{3.3}$$

But it is well known (see for example [3, p. 10] for $r = 1$) that the generating function for $v_r(k,m,n)$ is also (3.3). Hence we have (3.1). This identity is also evident from the Ferrers graph.

To illustrate (3.1) and (3.2) let $m = 7,\ n = 4,\ k = 3$ and $r = 2$. The sequences enumerated by $f_3(7,5;2,0)$ are $0,0,1,3,3$, $0,0,2,2,3$ and $0,1,1,2,3$. The function $u_2(3,7,4)$ counts the corresponding partitions, namely $13^2,\ 2^23$ and 1^223. The partitions which $v_2(3,7,4)$ enumerates are $2^23,\ 13^2$ and 124. From the graphs

. . .	. . .	. . . .
. .	. . .	. .
. .	.	.

we observe that 13^2 is the conjugate of 2^23, 2^23 is the conjugate of 13^2 and 1^223 is the conjugate of 124.

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3. P.A.M. MacMahon, *Combinatory Analysis,* Vol. 2, Cambridge, 1916.

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SPECIAL CASES

Putting $r = 1,\ s = 0,$ we obtain the generating function for the Fibonacci sequence (see [3] and Riordan [6]). Putting $r = 2,\ s = -1,$ we obtain the generating function for the Lucas sequence (see [3] and Carlitz [1]).

Other results in Riordan [6] carry over to the H-sequence. The H-sequence (and the Fibonacci and Lucas sequences), and the generalized Fibonacci and Lucas sequences are all special cases of the W-sequence studied by the author in [4]. More particularly,

$$\{H_n\} = \{w_n(r, r+s; 1, -1)\}$$

and so

$$\{f_n\} = \{w_n(1, 1; 1, -1)\}, \qquad \{a_n\} = \{w_n(2, 1; 1, -1)\} .$$

Interested readers might consult the article by Kolodner [5] which contains material somewhat similar to that in [3], though the methods of treatment are very different.

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