

ANTIMAGIC SQUARES DERIVED FROM THE THIRD-ORDER MAGIC SQUARE

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In a third-order antimagic square, the three elements along each of the eight lines—three rows, three columns, and two unbroken diagonals—have different sums. An antimagic square, its rotations and reflections are equivalent and count as only one square.

It is not difficult to modify the distribution of the digits around the central 5 of the unique nine-digit third-order magic square

$$\begin{matrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{matrix}$$

so as to convert it into the antimagic square

$$\begin{matrix} 1 & 2 & 3 \\ 6 & 5 & 8 \\ 9 & 4 & 7 \end{matrix}$$

while preserving an odd-even alternation of digits around the perimeter. Nor, to set up a sequence of antimagic squares,

$$\begin{matrix} 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 \\ 9 & 5 & 8 & 9 & 5 & 8 & 9 & 5 & 7 & 9 & 5 & 6 & 7 & 5 & 6 & 7 & 5 & 6 \\ 6 & 7 & 4 & 4 & 7 & 6 & 4 & 8 & 6 & 4 & 8 & 7 & 4 & 8 & 9 & 4 & 9 & 8 \end{matrix}$$

still around the central 5, in which each square results from the interchange of two digits in the previous square.

The *complements* of the squares in this sequence, obtained by subtracting each of the digits from 10, form the similar sequence

$$\begin{matrix} 9 & 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 \\ 1 & 5 & 2 & 1 & 5 & 2 & 1 & 5 & 3 & 1 & 5 & 4 & 3 & 5 & 4 & 3 & 5 & 4 \\ 4 & 3 & 6 & 6 & 3 & 4 & 6 & 2 & 4 & 6 & 2 & 3 & 6 & 2 & 1 & 6 & 1 & 2 \end{matrix}$$

Of course, if a square is antimagic, its complement also is. The eight sums of the complementary square may be obtained by subtracting each sum of the parent square from 30.

The question naturally arises, what is the minimum number of digits that need to change position and what is the minimum number of moves, or interchanges, necessary in order to convert the magic square into an antimagic square?

THE CRITERIA

To convert the magic square into an antimagic square by interchanging digits, two conditions must be met:

- (1) Not more than one line (sum = 15) can remain unaltered;
- (2) If two or more lines contain the same single changed element, only one of those lines can be left without another change.

There are 16 distinct ways in which three markers (x) can be distributed on a nine-cell 3-by-3 array. Thus

$$\begin{matrix} x00 & 0x0 & x0x & x0x & 0x0 & xxx & 000 & xx0 \\ 0x0 & 0x0 & 0x0 & 000 & x00 & 000 & xxx & x00 \\ 00x & x00 & 000 & 0x0 & 00x & 000 & 000 & 000 \\ \\ x00 & 0x0 & xx0 & xx0 & xx0 & xx0 & x0x & 0x0 \\ xx0 & xx0 & 00x & 000 & 000 & 000 & 000 & x0x \\ 000 & 000 & 000 & x00 & 0x0 & 00x & x00 & 000 \end{matrix}$$

If the x 's indicate the elements to be moved by interchange, then only the first five configurations meet the first condition, and none of those five meet the second condition. So at least four digits must be involved in the interchange.

There are 23 distinct ways in which four markers (x) can be distributed on a nine-cell 3-by-3 array. Thus

xxx	xxx	$x00$	$0x0$	$xx0$	$xx0$	$xx0$	$xx0$
$x00$	000	xxx	xxx	$xx0$	000	000	$x0x$
000	$0x0$	000	000	000	$xx0$	$0xx$	000
$x0x$	$xx0$	$xx0$	$xx0$	$0x0$	$x0x$	$x0x$	$0x0$
$x00$	$00x$	$00x$	$0x0$	$xx0$	$0x0$	000	$x0x$
$00x$	$x00$	$0x0$	$00x$	$00x$	$00x$	$x0x$	$0x0$
xxx	xxx	$x0x$	$xx0$	$xx0$	$xx0$	$x0x$	
$0x0$	000	$0x0$	$0xx$	000	$00x$	$0x0$	
000	$x00$	$x00$	000	$x0x$	$00x$	$0x0$	
A	B	C	D	E	F	G	

If the x 's indicate the elements to be moved in the interchanging, then only the last seven meet both conditions. Each of the three symmetrical arrangements (A , F and G) can be applied to the magic square in four ways, and the four asymmetrical configurations (B , C , D and E) and their mirror images can each be applied in four ways. So there are 44 applications of change patterns to consider.

TWO INTERCHANGES

Four elements can be divided into two pairs in three distinct ways. These pairings are applied to the seven change patterns. If both members of a pair fall on the same line, interchange of their positions does not affect the sum of the elements of that line. Each of the patterns A , C , D and E has one 000 line unaffected by interchange of the x 's. In any pairing of their x elements, any interchange between members of the pairs leaves the sum of the elements unchanged in one of the lines involved. Thus two lines of the square retain their original, and hence equal, sums.

In patterns F and G interchange between members of the pairs leaves the pairs in the original lines or interchanges the elements of a column and a row. Thus that column and row retain their original equal sums.

In the asymmetrical B , the interchange

$$\begin{array}{ccc} abc & cda \\ 000 & \rightarrow 000 \\ doo & boo \end{array}$$

leaves only one line sum unmodified. However, when applied to the nine-digit magic square in each of the eight possible ways, duplicate sums of 12, 14, 16 or 18 appear after the interchange.

Consequently, no antimagic square can be created by interchange of the members in each of two pairs of the elements of the nine-digit third-order magic square.

THREE INTERCHANGES

There are 24 permutations of the four elements M , N , P , Q . In 15 of these at least one of the elements has not moved from its original position. In 3 others, there have been two interchanges of positions. The other 6 can be attained from $MNPQ$ by three successive interchanges, namely:

$$\begin{array}{ll} U - NPQM : MN, MP, MQ & X - PQNM : MP, NQ, MN \\ V - NQMP : MN, MP, PQ & Y - QMNP : PQ, MQ, NM \\ W - PMQN : MP, NM, NQ & Z - QPMN : MQ, NP, NM \end{array}$$

These six permutations, identified by the prefaced letters U , V , W , X , Y , Z can be applied to each of the seven patterns A , B , C , D , E , F , G in their various orientations on the magic square. The letters M , N , P , Q may be arbitrarily assigned to the four x elements of the pattern without affecting the ultimate position of the interchanged digits.

Pattern B is asymmetrical, so it and its mirror image both are applied to the magic square in the four orientations, proceeding clockwise around the square. Thus the magic square operated on by pattern B becomes

M	N	P	Q	1	M	8	1	Q	P	1	6
3	5	7	3	5	N	3	5	7	N	5	7
Q	9	2	4	9	P	P	N	M	M	9	Q
B_1				B_2			B_3				B_4

<i>P</i>	<i>N</i>	<i>M</i>	<i>8</i>	<i>1</i>	<i>P</i>	<i>Q</i>	<i>1</i>	<i>6</i>	<i>M</i>	<i>1</i>	<i>Q</i>
3	5	7	3	5	<i>N</i>	3	5	7	<i>N</i>	5	7
4	9	<i>Q</i>	<i>Q</i>	9	<i>M</i>	<i>M</i>	<i>N</i>	<i>P</i>	<i>P</i>	9	2
	<i>B</i> ₅			<i>B</i> ₆		<i>B</i> ₇			<i>B</i> ₈		

The particular orientation of the operating pattern is indicated by the numerical subscript. This notation will be followed with subsequent pattern operators.

In *B*₁, *M* = 8, *N* = 1, *P* = 6, *Q* = 4, so for the six permutations we have:

<i>U</i> - <i>N P Q M</i>	<i>V</i> - <i>N Q M P</i>	<i>W</i> - <i>P M Q N</i>
1 6 4 8	1 4 8 6	6 8 4 1
<i>X</i> - <i>P Q M N</i>	<i>Y</i> - <i>Q M N P</i>	<i>Z</i> - <i>Q P M N</i>
6 4 1 8	4 8 1 6	4 6 8 1

The digits in each of the permutations are placed, in sequential order, in the *M, N, P, Q* positions of the square array, *B*₁. Not all of the permutations will yield antimagic squares. No sum of 15 remains, but other duplicate sums may appear in the process. In *B*₁, no antimagic squares are produced by *V, W,* and *Y*. Indeed, *V* does not produce an antimagic square in any *B*_{*i*}.

When the pattern orientations are 180° apart, as in *B*₁*B*₃, *B*₂*B*₄, *B*₅*B*₇, and *B*₆*B*₈, complementary antimagic squares are produced. (The *MNPQ* sets are complementary.) Only one of each complementary pair is recorded below in identifying the twenty antimagic squares produced by pattern *B*. In general, the orientation of the pattern which has operated on the magic square can be identified by the digits which occupy the same position in the antimagic square as they did in the original magic square.

1 6 4	6 4 1	4 6 8	6 1 7	6 1 2
3 5 7	3 5 7	3 5 7	3 5 2	3 5 8
8 9 2	8 9 2	1 9 2	4 9 8	4 9 7
8 1 4	2 8 1	1 2 8	1 6 2	6 8 2
3 5 2	3 5 7	3 5 7	3 5 7	3 5 7
7 9 6	4 9 6	4 9 6	4 9 8	4 9 1

Since pattern *B* always leaves a mid-row or mid-column undisturbed, each of these antimagic squares has a central 5. The last square is particularly noteworthy in that its sums are 10, 11, 12, 13, 14, 15, 16, and 22, where seven of the eight are consecutive numbers. No antimagic square has all eight sums in arithmetic progression. [1,2]

In *Pattern C*,

<i>MNo</i>
<i>oPo</i>
<i>Qoo</i>

application of *X* or *Z* leaves the sums of the row and diagonal which have the upper right-hand element in common unchanged and hence equal. *U, V, W,* and *Y* also fail to produce an antimagic square in the eight orientations of *C*.

In *Pattern D*,

<i>MNo</i>
<i>oPQ</i>
<i>ooo</i>

application of *X* or *Z* fails to produce an antimagic square in any of the eight orientations. Nor do any of the permutations produce one with *D*₂ or *D*₄. As with pattern *B*, complementary antimagic squares are produced by pattern orientations 180° apart. One of each of the eight complementary pairs resulting from pattern *D* is recorded below:

1 5 6	1 7 6	5 8 6	7 8 6
3 7 8	3 8 5	3 7 1	3 1 5
4 9 2	4 9 2	4 9 2	4 9 2
8 5 6	8 7 6	8 5 6	8 5 1
3 2 1	3 1 2	3 7 2	6 3 7
4 9 7	4 9 5	4 9 1	4 9 2

Pattern E fails to yield any antimagic squares when X and Z are used as operators, nor from E_2 or E_4 with any operator. As in B and D , the antimagic squares formed are in complementary pairs. One member of each of the eight pairs from pattern E follows:

1 4 6	1 2 6	4 8 6	2 8 6
3 5 7	3 5 7	3 5 7	3 5 7
2 9 8	8 9 4	2 9 1	1 9 4
8 6 2	2 1 6	7 1 6	4 1 6
3 5 7	3 5 4	3 5 2	3 5 2
1 9 4	8 9 7	8 9 4	7 9 8

In common with the squares from B , all these squares have 5 as a central digit.

Pattern A,

MNP
oQo ,
ooo

being symmetrical, can be applied to the magic square in only four orientations. Of the permutations, only X fails to produce an antimagic square from some orientation. Those formed when the pattern orientations are 180° apart are complements. One of each of the six complementary pairs is given here:

1 6 5	1 5 8	6 8 5	5 8 1	5 6 8	8 1 5
3 8 7	3 6 7	3 1 7	3 6 7	3 1 7	3 7 2
4 9 2	4 9 2	4 9 2	4 9 2	4 9 2	4 9 6

The sums of the fourth square in this set are 10, 11, 12, 13, 14, 15, 16, and 23, another case where seven of the sums are consecutive integers.

In *patterns F and G*, the other two symmetrical patterns, X and Z merely interchange a row and column and leave the sums equal. The other four permutations fail to produce an antimagic square with any of the four orientations.

SUMMARY

In order to convert the nine-digit third-order magic square into an antimagic square by interchange of digits, not less than four digits must be moved in three successive interchanges. The four digits must fall into one of four basic patterns (B , D , E , A) to which one of the six permutations U , V , W , X , Y , Z is applied. The 64 antimagic squares which can be produced in this manner fall into 32 complementary pairs. Complementary pairs are produced by patterns 180° apart in orientation. Six of these pairs come from a symmetrical pattern. The 26 pairs that are the result of applying asymmetrical patterns are produced in equal quantities by the patterns and their mirror images. The central digit of 36 of the antimagic squares is 5. The frequency of occurrence of the other central digits follows each of the following digits in parentheses: 1(4), 2(3), 3(5), 4(2), 6(2), 7(5), 8(3), 9(4). Two of the squares have seven of their sums in arithmetic progression, with $d = 1$.

REFERENCES

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