ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A.P. HILLMAN
University of New Mexico, Albuquerque, New Mexico 87131

Send all communications regarding Elementary Problems and Solutions to Professor A.P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-298 Proposed by Richard Blazej, Queens Village, New York.

Show that

$$5F_{2n+3} \cdot F_{2n-3} = L_{4n} + 18.$$

B-299 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Establish a simple closed form for

$$F_{2n+3} - \sum_{k=1}^{n} (n+2-k)F_{2k}.$$

B-300 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Establish a simple closed form for

$$L_{2n+2} - \sum_{k=1}^{n} (n+3-k)F_{2k}.$$

B-301 Proposed by Phil Mana, Albuquerque, New Mexico.

Let $[x]$ denote the greatest integer in x, i.e., the integer m with $m \leqslant x < m+1$. Also let

$$A(n) = (n^2 + 6n + 12)/12 \qquad \text{and} \qquad B(n) = (n^2 + 7n + 12)/6.$$

Does

$$[A(n)] + [A(n+1)] = [B(n)]$$

for all integers n? Explain.

B-302 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Prove that $F_n - 1$ is a composite integer for $n \geqslant 7$ and that $F_n + 1$ is composite for $n \geqslant 4$.

B-303 Proposed by David Singmaster, Polytechnic of the South Bank, London, England.

In B-260, it was shown that

$$\sigma(mn) > \sigma(m) + \sigma(n) ,$$

where $\sigma(n)$ is the sum of the positive integral divisors of n. What relation holds between $\sigma(mn)$ and $\sigma(m)\sigma(n)$?

SOLUTIONS

3 SYMBOL GOLDEN MEAN

B-274 Proposed by C.B.A. Peck, State College, Pennsylvania.

Approximate $(\sqrt{5} - 1)/2$ to within 0.002 using at most three distinct familiar symbols. (Each symbol may represent a number or an operation and may be repeated in the expression.)

I. Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

We may use the well-known continued fraction expansion for

$$\theta = \tfrac{1}{2}(\sqrt{5} - 1) : \qquad \theta = 1/1 + 1/1 + 1/+ \cdots$$

with convergents:

$$0/1, \quad 1/1, \quad 1/2, \quad 2/3, \quad 3/5, \quad 5/8, \quad 8/13, \quad 13/21, \cdots .$$

Clearly, any such expression satisfies the conditions of the problem, since it uses only the three symbols "1," "+" and "/" (or "_," for the last symbol, representing division). To obtain any desired degree of accuracy, we may use the inequality:

$$|\theta - p_n/q_n| < 1/q_nq_{n+1} ,$$

where p_n/q_n is the n^{th} convergent of the continued fraction. For this problem, we desire $1/q_nq_{n+1}$ to be less than $.002$, i.e., q_nq_{n+1} must exceed 500. Now

$$13 \cdot 21 = 273 < 500, \quad \text{while} \quad 21 \cdot 34 = 714 > 500 ,$$

so we may take the continued fraction expression for $13/21$ as one solution (the simplest solution), although the corresponding expression for any higher convergent is also a solution.

II. The Proposer gave the solution in I and also noted that

$$(\sqrt{5} - 1)/2 \doteq \pi^2/2^2 \doteq 0.6169$$

is easily obtained from

$$\pi \doteq \sqrt{8(\sqrt{5} - 1)} ,$$

given in P. Poulet, *C'est Encore* π, Sphinx, Vol. 6, No. 12, Dec. 1936, pp. 208–212.

TWO IN ONE

B-275 Proposed by Warren Cheves, Littleton, North Carolina.

Show that

$$F_{mn} = L_mF_{m(n-1)} + (-1)^{m+1}F_{m(n-2)} .$$

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

The required equation is a condensation into one identity of the two identities (I_{21}) and (I_{23}) on page 59 of Hoggatt's book, *Fibonacci and Lucas Numbers, viz.,*

$$F_{n+p} + F_{n-p} = F_nL_p , \; p \text{ even},$$
$$F_{n+p} - F_{n-p} = F_nL_p , \; p \text{ odd}.$$

In these two equations, replace n by $mn - m$ and p by m.

Also solved by Paul S. Bruckman, Wray G. Brady, Herta T. Freitag, John W. Milsom, C.B.A. Peck, A.G. Shannon (New South Wales), and the Proposer.

ONLY TWO SOLUTIONS

B-276 Proposed by Graham Lord, Temple University, Philadelphia, Pennsylvania.

Find all the triples of positive integers m, n, and x such that

$$F_h = x^m, \quad \text{where} \quad h = 2^n \text{ and } m > 1.$$

Solution by Phil Tracy, Lexington, Massachusetts.

It has only the trivial solutions $n = 0$ and $n = 1$ since F_{2n} is an integral multiple of 3 but not of 9 when $n > 1$. One can see this as follows. Modulo 9, the Fibonacci numbers repeat in blocks of 24. Examining the block, one finds $3|F_m$ if and only if $4|m$ while $9|F_m$ if and only if $12|m$. Finally, 2^n is an integral multiple of 4 but not of 12, when $n > 1$.

Also solved by Paul S. Bruckman, Herta T. Freitag, and the Proposer.

A LUCAS-FIBONACCI CONGRUENCE

B-277 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $L_{2n(2k+1)} \equiv L_{2n} \pmod{F_{2n}}$.

Solution by David Zeitlin, Minneapolis, Minnesota.

Using the Binet formulas

$$F_n = (a^n - b^n)/(a - b) \quad \text{and} \quad L_n = a^n + b^n,$$

one easily shows that

$$(1) \qquad L_{m+p} - L_{m-p} = 5F_mF_p, \quad p \text{ even}.$$

Set $m = 2n(k+1)$ and $p = 2nk$ in (1) to obtain

$$L_{2n(2k+1)} - L_{2n} = 5F_{2n(k+1)}F_{2nk}.$$

Since $F_{2n}|F_{2nk}$, the result follows.

REMARK. Since $F_{2n}|F_{2n(k+1)}$, the result can be stronger, i.e.,

$$L_{2n(2k+1)} \equiv L_{2n} \pmod{F_{2n}^2}.$$

Also solved by Gregory Wulczyn and the Proposer.

ANOTHER LUCAS-FIBONACCI CONGRUENCE

B-278 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Prove that $L_{(2n+1)(4k+1)} \equiv L_{2n+1} \pmod{F_{2n+1}}$.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

$$(1) \qquad L_{(2n+1)(4k+1)} - L_{2n+1} = a^{(2n+1)(4k+1)} - b^{(2n+1)(4k+1)} - b^{2n+1}.$$

The quotient of (1) by

$$\frac{a^{2n+1} - b^{2n+1}}{\sqrt{5}} = 5\left[\frac{a^{4n+2} - b^{4n+2}}{\sqrt{5}} + \frac{a^{2(4n+2)} - b^{2(4n+2)}}{\sqrt{5}} + \cdots + \frac{a^{4k(2n+1)} - b^{4k(2n+1)}}{\sqrt{5}}\right]$$

$$= 5(F_{4n+2} + F_{4(2n+1)} + \cdots + F_{4k(2n+1)}),$$

an integer.

Also solved by David Zeitlin and the Proposer.

CORRECTED AND REINSERTED

Due to the typographical error in the original statement of B-279, the deadline for receipt of solutions has been extended. The error was corrected and the correct problem solved by Paul S. Bruckman, Charles Chouteau, Edwin T. Hoefer, and the Proposer. The error was also noted by Wray G. Brady. The corrected version is:

B-279 Find a closed form for the coefficient of x^n in the Maclaurin series expansion of $(x + 2x^2)/(1 - x - x^2)^2$.

****