SOME INTERESTING NECESSARY CONDITIONS FOR $(a-1)^n + (b-1)^n - (c-1)^n = 0$

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In the present note we obtain certain inequalities which are necessary for the equation of the title to hold for positive integral n and real $a, b,$ and c satisfying $1 < a \leqslant b < c,$ and illustrate with several examples. Several preliminary lemmas are required.

Lemma 1. $(a-1)^x + (b-1)^x - (c-1)^x$ vanishes at $x = n$ if and only if

$$a^x + b^x - c^x = P_{n-1}(x)$$

at $x = 0, 1, \cdots, n,$ where $P_{n-1}(x)$ is a polynomial of degree $n-1$.

Proof. Apply the n^{th} order difference operator Δ^n to $a^x + b^x - c^x$ to obtain

$$\Delta^n(a^x + b^x - c^x) = (a-1)^n a^x + (b-1)^n b^x - (c-1)^n c^x ,$$

which vanishes at $x = 0$ if and only if $a^x + b^x - c^x$ behaves as a polynomial of degree $n-1$ at $x = 0, 1, \cdots, n.$

A result in Pólya and Szegö [1] is needed for the next lemma and may be stated as follows for present purposes: If $a < b < c$ and $\mu_1, \mu_2,$ and μ_3 are positive, then

$$\mu_1 a^x + \mu_2 b^x - \mu_3 c^x$$

has exactly one real simple zero. As an immediate consequence of this and other elementary considerations we have the following result.

Lemma 2. Let

$$f(x) = a^x + b^x - c^x ,$$

where $1 < a \leqslant b < c.$ Then $f^{(k)}(x)$ has exactly one real simple zero, one stationary point at which $f^{(k)}$ has a positive maximum and to the right of which $f^{(k)}$ is monotone decreasing.

In the following we will always let $f(x)$ and $P_{n-1}(x)$ be as stated in Lemmas 1 and 2.

Lemma 1 says that

$$F(x) \equiv f(x) - P_{n-1}(x)$$

has at least $n+1$ zeros. That this is the exact number is assured by the next result.

Lemma 3. $F(x) \equiv f(x) - P_{n-1}(x)$ has at most $n+1$ zeros (counting multiplicity).

Proof. Assume that F has at least $n+2$ zeros. Then $F^{(n)}$ has at least 2 zeros. Since $P_{n-1}^{(n)} \equiv 0$ this implies that $f^{(n)}$ has 2 zeros in contradiction to Lemma 2.

Write

$$P_{n-1}(x) = c_1 + c_2 x + \cdots + c_n x^{n-1} .$$

Our final preliminary result may be stated as follows.

Lemma 4. $c_n > 0.$

Proof. We know that

$$f(x) - P_{n-1}(x) = 0$$

at the $n+1$ points $x = 0, 1, \cdots, n.$ Thus

$$f^{(n-1)}(x) = (n-1)! c_n$$

at two points which because of Lemma 2 implies that c_n is positive.

Now consider the special case when $n = 2$.

Theorem 1. If $(a-1)^2+(b-1)^2-(c-1)^2 = 0$ then

(1) $$ab/c < e^{a+b-c-1},$$

(2) $$a^a b^b/c^c > e^{a+b-c-1},$$

and

(3) $$a^{a^2} b^{b^2}/c^{c^2} < e^{a+b-c-1}.$$

Proof. By the preceding lemmas we know that in $P_1(x) = c_1 + c_2 x$ we have $c_2 > 0$, that

$$f(x) = a^x + b^x - c^x$$

is monotone decreasing for all sufficiently large x, and that $f(x) - P_1(x)$ has simple zeros at precisely $x = 0, 1, 2$. This requires that $f'(2) < P_1'(2)$ and in turn $f'(1) > P_1'(1)$ and $f'(0) < P_1'(0)$. In other words, using the last of the three inequalities, we have $\ln(ab/c) < c_2$. c_2 can be easily determined from the coincidence of $f(x)$ and $P_1(x)$ at $x = 0, 1, 2$ to give $c_2 = a + b - c - 1$. Hence, finally, $ab/c < e^{a+b-c-1}$. The inequalities (2) and (3) follow in a similar manner from $f'(1) > P_1'(1)$ and $f'(2) < P_1'(2)$.

For the case of $n = 3$, the following result can be obtained by arguments similar to those used above for Theorem 1. The proof is therefore omitted.

Theorem 2. If $(a-1)^3+(b-1)^3-(c-1)^3 = 0$, then

(1) $$ab/c > e^{a+b-c-1-c_3},$$

(2) $$a^a b^b/c^c < e^{a+b-c-1+c_3},$$

(3) $$a^{a^2} b^{b^2}/c^{c^2} > e^{a+b-c-1+3c_3},$$

and

(4) $$a^{a^3} b^{b^3}/c^{c^3} < e^{a+b-c-1+5c_3},$$

where

$$c_3 = \tfrac{1}{2}[a^2+b^2-c^2+1-2a-2b+2c].$$

Inequalities of a similar nature may be found for any given value of n, however let us proceed to a result for arbitrary n. By $L_n(a)$ we shall mean the partial sum of the first $n-1$ terms of the formal Maclaurin series for $\log a$, i.e.,

$$L_n(a) = \sum_{k=1}^{n-1} (-1)^{k+1} \frac{a^k}{k}.$$

Theorem 3. Let $(a-1)^n + (b-1)^n - (c-1)^n = 0$. Then

$$(-1)^n(\log a + \log b - \log c) < (-1)^n[L_n(a) + L_n(b) - L_n(c)].$$

Proof. Proceeding as for Theorem 1, we find that

$$(-1)^n f'(0) < (-1)^n P_{n-1}'(0).$$

Write

$$P_{n-1}(x) = \sum_{k=0}^{n-1} c_k x^{(k)},$$

where

$$x^{(k)} = x(x-1)\cdots(x-n+1).$$

Gregory-Newton interpolation gives

$$c_k = \Delta^k f(0)/k!.$$

Now

$$\Delta^k a^x = (a-1)^k a^x,$$

from which it follows that

$$\Delta^k f(0) = (a-1)^k + (b-1)^k - (c-1)^k.$$

Therefore, since

$$\frac{d}{dx}x^{(k)}\bigg|_{x=0} = (-1)^{k-1}(k-1)!\,,$$

we have

$$(-1)^n(\ln a + \ln b - \ln c) < (-1)^n \sum_{k=1}^{n-1} (-1)^{k+1} \frac{(k-1)!}{k!} [(a-1)^k + (b-1)^k - (c-1)^k]\ ,$$

as desired.

We give an indication, in the following examples, of the sharpness of the inequalities obtained above. First we take $n = 2$, $a = 4$, $b = 5$, in which case inequalities (2) and (3) of Theorem 1 yield $c < 6.5$ and $c > 5.9$, respectively, bracketing the known solution $c = 6$. This example corresponds to the well-known Pythagorean triple $3,4,5$ which satisfies $3^2 + 4^2 = 5^2$. If we now take $n = 3$, $a = 2$, $b = 3$, then inequalities (2) and (4) of Theorem 2 give $c < 3.2$ and $c > 3$, whereas the actual solution of

$$1 + 2^3 - (c-1)^3 = 0$$

is

$$c = 1 + \sqrt[3]{9} \cong 3.08.$$

The sharpness of these results seems rather surprising when one considers that they are based on such simple considerations as the relative slope of two curves at their points of intersection.

REFERENCE

1. G. Pólya and G. Szegö, *Aufgahen und Lehrsätze aus der Analysis,* Berlin, 1925.

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