# ADVANCED PROBLEMS AND SOLUTIONS

Edited by
**RAYMOND E. WHITNEY**
**Lock Haven State College, Lock Haven, Pennsylvania 17745**

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within to months after publication of the problem.

*H-249 Proposed by F. D. Parker, St. Lawrence University, Canton, New York.*

Find an explicit formula for the coefficients of the Maclaurin series for

$$\frac{b_o + b_1x + \cdots + b_kx^k}{1 + \alpha x + \beta x^2}\;.$$

*H-250 Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Show that if

$$A(n)F_{n+1} + B(n)F_n = C(n) \qquad (n = 0, 1, 2, \cdots),$$

where the $F_n$ are the Fibonacci numbers and $A(n), B(n), C(n)$ are polynomials, then

$$A(n) \equiv B(n) \equiv C(n) \equiv 0.$$

*H-251 Proposed by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.*

Prove the identity:

$$\sum_{n=0}^{\infty} \frac{x^{n^2}}{[(x)_n]^2} = \sum_{n=0}^{\infty} \frac{x^n}{(x)_n}\,,$$

where

$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n), \quad (x)_0 = 1\;.$$

## SOLUTIONS

## SOME SUM

*H-219 Proposed by Paul Bruckman, University of Illinois, Urbana, Illinois.*

Prove the identity

$$(-1)^n \binom{x}{n} \sum_{i=0}^{n} \binom{n}{i}(-2)^i \cdot \frac{x-n}{x-i} = \sum_{i=0}^{n} \binom{x}{i}\quad ,$$

where

$$\binom{x}{i} = \frac{x(x-1)(x-2)\cdots(x-i+1)}{i!}$$

(x) not necessarily an integer.

*Solution and generalization by H. Gould, West Virginia University, Morgantown, West Virginia.*

We shall obtain the slightly more general formula

$$(1)\qquad (-1)^n\binom{x}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+t)^k\,\frac{x-n}{x-k}=\sum_{k=0}^{n}\binom{x}{k}t^k\,.$$

Examination of Bruckman's formula suggests that the formula can be found from the partial fraction expansion

$$(2)\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x}{x+k}=\binom{x+n}{n}^{-1},$$

which is formula (1.41) in my book, *Combinatorial Identities* (a standardized set of tables listing 500 binomial coefficient summations, revised edition, published by the author, Morgantown, W. Va., 1972). This is a familiar and well-known formula. Besides (2) we shall need below the formula

$$\binom{-x}{k}=(-1)^k\binom{x+k-1}{k},$$

the binomial theorem, and simple operations on series.

We make a straightforward attack on the left-hand side of (1) and find

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+t)^k\,\frac{x-n}{x-k}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x-n}{x-k}\sum_{j=0}^{k}\binom{k}{j}t^j$$

$$=\sum_{j=0}^{n}\binom{n}{j}t^j\sum_{k=j}^{n}(-1)^k\binom{n-j}{k-j}\frac{x-n}{x-k}=\sum_{j=0}^{n}(-1)^j\binom{n}{j}t^j\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{x-n}{x-j-k}$$

$$=\sum_{j=0}^{n}(-1)^j\binom{n}{j}t^j\,\frac{n-x}{j-x}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{-x+j}{-x+j+k}$$

$$=\sum_{j=0}^{n}(-1)^j\binom{n}{j}t^j\,\frac{n-x}{j-x}\binom{n-x}{n-j}^{-1},\ \text{by (2)},$$

$$=\sum_{j=0}^{n}(-1)^j\binom{n}{j}t^j\binom{n-x-1}{n-j}^{-1}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}t^{n-j}\binom{n-x-1}{j}^{-1}$$

Therefore,

$$(-1)^n\binom{x}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+t)^k\,\frac{x-n}{x-k}=\binom{x}{n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}t^{n-j}\binom{n-x-1}{j}^{-1}$$

$$=\binom{x}{n}\binom{n-x-1}{n}^{-1}\sum_{j=0}^{n}(-1)^j\binom{n-x-1-j}{n-j}t^{n-j}=\binom{x}{n}(-1)^n\binom{x}{n}^{-1}\sum_{j=0}^{n}(-1)^j(-1)^{n-j}\binom{x}{n-j}t^{n-j}$$

$$=\sum_{j=0}^{n}\binom{x}{n-j}t^{n-j}=\sum_{j=0}^{n}\binom{x}{j}t^j\,,$$

as desired to show. Bruckman's formula (1) occurs when $t=1$, and formula (2) occurs when $t=0$. Thus (2) is not only used to prove (1) but is a special case of it.

We may rewrite (1) in the form

$$(3)\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1+t)^k}{x-k}=(-1)^n\,\frac{1}{x-n}\binom{x}{n}^{-1}\sum_{j=0}^{n}\binom{x}{j}t^j\,.$$

Recall the simple, well known inversion pair

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(k)$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k)\,,$$

and we see that (3) inverts to give

$$\sum_{k=0}^{n} \binom{n}{k} \frac{x-n}{x-k} \binom{x}{k}^{-1} \sum_{j=0}^{k} \binom{x}{j} t^j = (1+t)^n\,. \tag{4}$$

Now, however, the power series expansion of $(1+t)^n$ is unique, so that the coefficient of $t^j$ in (4) must be precisely $\binom{n}{j}$, so that we have evidently proved

$$\binom{x}{j} \sum_{k=j}^{n} \binom{n}{k}\binom{x}{k}^{-1} \frac{x-n}{x-k} = \binom{n}{j} \tag{5}$$

for all real $x$. This formula is actually just a special case of (4.1) in *Combinatorial Identities* which occurs when we set $z = n$ there and replace $x$ by $x - 1$. However (5) is an interesting way to express this case.

Many other interesting sums can be found from (1). Thus by taking $r^{th}$ derivatives we have at once the identity

$$\binom{x}{n} \sum_{k=r}^{n} (-1)^{n-k} \binom{n}{k}\binom{k}{r} (1+t)^{k-r} \frac{x-n}{x-k} = \sum_{k=r}^{n} \binom{x}{k}\binom{k}{r} t^{k-r}\,, \tag{6}$$

which will express other relations in *Combinatorial Identities* in different ways. For $t = 0$, Eq. (6) yields nothing more than a variant of (2) again.

[See also Paul Bruckman, Problem H-219, *The Fibonacci Quarterly*, Vol. 11, No. 2 (April 1973), p. 185.]

*Also solved by G. Lord, P. Tracy, L. Carlitz, and the Proposer.*

## ON Q

*H-220 Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Show that

$$\sum_{k=0}^{\infty} \frac{a^k z^k}{(z)_{k+1}} = \sum_{r=0}^{\infty} \frac{a^r q^{r^2} z^{2r}}{(z)_{r+1}(az)_{r+1}}\,,$$

where

$$(z)_n = (1-z)(1-qz)\cdots(1-q^{n-1}z), \qquad (z)_0 = 1\,.$$

*Solution by the Proposer.*

It is well known that

$$\frac{1}{(z)_{k+1}} = \sum_{r=0}^{\infty} \begin{bmatrix} k+r \\ r \end{bmatrix} z^r\,,$$

where

$$\begin{bmatrix} k+r \\ r \end{bmatrix} = \frac{(q)_{k+r}}{(q)_k (q)_r} = \begin{bmatrix} k+r \\ k \end{bmatrix}.$$

Thus

$$\sum_{k=0}^{\infty} \frac{a^k z^k}{(z)_{k+1}} = \sum_{k=0}^{\infty} a^k z^k \sum_{r=0}^{\infty} \begin{bmatrix} k+r \\ r \end{bmatrix} z^r = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^k .$$

On the other hand,

$$\sum_{r=0}^{\infty} \frac{a^r q^{r^2} z^{2r}}{(z)_{r+1}(az)_{r+1}} = \sum_{r=0}^{\infty} a^r q^{r^2} z^{2r} \sum_{s=0}^{\infty} \begin{bmatrix} r+s \\ s \end{bmatrix} z^s \sum_{j=0}^{\infty} \begin{bmatrix} r+j \\ j \end{bmatrix} a^j z^j$$

$$= \sum_{n=0}^{\infty} z^n \sum_{2r+s+j=n} \begin{bmatrix} r+s \\ s \end{bmatrix} \begin{bmatrix} r+j \\ j \end{bmatrix} a^{r+j} q^{r^2} = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} a^k \sum_{r=0}^{k} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} n-k \\ r \end{bmatrix} q^{r^2} .$$

Hence it remains to show that

$$\sum_{r=0}^{k} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} n-k \\ r \end{bmatrix} q^{r^2} = \begin{bmatrix} n \\ k \end{bmatrix}$$

or what is the same thing

(*)
$$\sum_{r=0}^{k} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} q^{r^2} = \begin{bmatrix} m+n \\ m \end{bmatrix} .$$

This can be proved rapidly as follows. We recall that

$$(1+z)(1+qz)\cdots(1+q^{n-1}z) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} z^r .$$

Then

$$\sum_{k=0}^{m+n} \begin{bmatrix} m+n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} z^k = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} (q^n z)^r \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}s(s-1)} z^s .$$

The coefficient of $z^n$ on the right is equal to

$$\sum_{r+s=n} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1)+nr+\frac{1}{2}s(s-1)} = \sum_{r=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)+nr+\frac{1}{2}(n-r)(n-r-1)}$$

$$= q^{\frac{1}{2}n(n-1)} \sum_{r=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} q^{r^2} .$$

This proves (*).

## CONGRUENCE FOR $F_n$ AND $L_n$

*H-221 Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Let $p = 2m + 1$ be an odd prime, $p \neq 5$. Show that if $m$ is even then

$$F_m \equiv 0 \pmod p \quad \left( \left( \frac{5}{p} \right) = +1 \right)$$

$$F_{m+1} \equiv 0 \pmod p \quad \left( \left( \frac{5}{p} \right) = -1 \right) \quad ;$$

If $m$ is odd then

$$L_m \equiv 0 \pmod p \quad \left( \left( \frac{5}{p} \right) = +1 \right)$$

$$L_{m+1} \equiv 0 \pmod p \quad \left( \left( \frac{5}{p} \right) = -1 \right) \quad ,$$

where $\left(\frac{5}{p}\right)$ is the Legendre symbol.

*Solution by the Proposer.*

Put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}\,, \qquad L_n = \alpha^n + \beta^n\,,$$

where $\alpha + \beta = 1,\ \alpha\beta = -1.$

Recall the identities

$$(*) \qquad L_{2m+1} - 1 = \begin{cases} 5F_mF_{m+1} & (m \text{ even}) \\ L_mL_{m+1} & (m \text{ odd}) \end{cases}.$$

Since $L_p \equiv 1 \pmod p$, it follows that

$$\begin{cases} F_mF_{m+1} \equiv 0 \pmod p & (m \text{ even}) \\ L_mL_{m+1} \equiv 0 \pmod p & (m \text{ odd}) \end{cases}.$$

1. Let $m$ be even. Since $(F_m, F_{m+1}) = 1$, it follows that either $F_m$ or $F_{m+1} \equiv 0 \pmod p$ but not both. Since

$$F_m \equiv 0 \pmod p \rightleftarrows \alpha^{2m} \equiv 1 \pmod p$$

and

$$F_{m+1} \equiv 0 \pmod p \rightleftarrows \alpha^{2m+2} \equiv -1 \pmod p,$$

we must show that

$$\begin{cases} \alpha^{p-1} \equiv 1 \pmod p & \left(\left(\frac{5}{p}\right) = +1\right) \\ \alpha^{p+1} \equiv -1 \pmod p & \left(\left(\frac{5}{p}\right) = -1\right) \end{cases}.$$

Now when $\left(\frac{5}{p}\right) = +1,\ p = \pi\pi',$ where $\pi, \pi'$ are primes in the quadratic field $Q(\sqrt{5}\,)$. Since

$$N(\pi) = N(\pi') = p$$

and $\alpha$ is a unit of the field we have

$$\alpha^{p-1} \equiv 1(\pi), \qquad \alpha^{p-1} \equiv 1(\pi')$$

and therefore $\alpha^{p-1} \equiv 1 \pmod p$.

On the other hand if $\left(\frac{5}{p}\right) = -1$, $p$ remains a prime in $Q(\sqrt{5}\,)$. Since

$$\alpha^p = \left(\frac{1+\sqrt{5}}{2}\right)^p \equiv \frac{1+5^{\frac{1}{2}(p-1)}\sqrt{p}}{2} \equiv \frac{1-\sqrt{p}}{2} \pmod p,$$

it is clear that $\alpha^p \equiv \beta \pmod p$, so that $\alpha^{p+1} \equiv \alpha\beta \equiv -1 \pmod p$.

2. Now let $m$ be odd. Since $(L_m, L_{m+1}) = 1$, it follows from (*) that either $L_m$ or $L_{m+1} \equiv 0 \pmod p$ but not both. Since

$$L_m \equiv 0 \pmod p \rightleftarrows \alpha^{2m} \equiv 1 \pmod p$$

and

$$L_{m+1} \equiv 0 \pmod p \rightleftarrows \alpha^{2m+2} \equiv -1 \pmod p,$$

it suffices to show that

$$\begin{cases} \alpha^{p-1} \equiv 1 \pmod p & \left(\left(\frac{5}{p}\right) = +1\right) \\ \alpha^{p+1} \equiv -1 \pmod p & \left(\left(\frac{5}{p}\right) = -1\right) \end{cases}.$$

However the proof of these congruences for $m$ even applies also when $m$ is odd.

This completes the proof.

REMARK. We have incidentally proved that

$$\begin{cases} \alpha^{p-1} \equiv 1 \pmod p & \left(\left(\frac{5}{p}\right) = +1\right) \\ \alpha^{p+1} \equiv 1 \pmod p & \left(\left(\frac{5}{p}\right) = -1\right) \end{cases}.$$

The first of these congruences is immediate but the second is less obvious.

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