IDENTITIES RELATING THE NUMBER OF PARTITIONS INTO AN EVEN AND ODD NUMBER OF PARTS

H. L. ALDER
University of California, Davis, California 95616
and
AMIN A. MUWAFI
American University of Beirut

1. INTRODUCTION

If $i \geqslant 0$ and $n \geqslant 1$, let $q_i^e(n)$ denote the number of partitions of n into an even number of parts, where each part occurs at most i times and let $q_i^o(n)$ denote the number of partitions of n into an odd number of parts, where each part occurs at most i times. If $i \geqslant 0$, let $q_i^e(0) = 1$ and $q_i^o(0) = 0$. For $i \geqslant 0$ and $n \geqslant 0$, let $\Delta_i(n) = q_i^e(n) - q_i^o(n)$.

For $i = 1$, it is well known [1] that

$$\Delta_1(n) = \begin{cases} (-1)^j & \text{if } n = \tfrac{1}{2}(3j^2 \pm j) \text{ for some } j = 0, 1, 2, \cdots, \\ 0 & \text{otherwise.} \end{cases}$$

For $i = 3$, Dean R. Hickerson [2] has proved that

$$\Delta_3(n) = \begin{cases} (-1)^n & \text{if } n = \tfrac{1}{2}(j^2 + j) \text{ for some } j = 0, 1, 2, \cdots, \\ 0 & \text{otherwise.} \end{cases}$$

For i an even number, Hickerson [2] has proved that

$$\Delta_i(n) = (-1)^n p_i^d(n),$$

where $p_i^d(n)$ is the number of partitions of n into distinct odd parts which are not divisible by $i + 1$ and $p_i^d(0) = 1$.

In this paper, we obtain formulae for $\Delta_i(n)$ for $i = 5$ and 7 in terms of the number of partitions into distinct parts taken from certain sets. These formulae, like those above, will allow rapid calculation of $\Delta_i(n)$ even for large values of n without the need to determine either $q_i^e(n)$ or $q_i^o(n)$. They will also allow verification of a conjecture by Hickerson [3] that, for $i = 5$ and 7, $\Delta_i(n)$ is nonnegative if n is even and nonpositive if n is odd.

2. THEOREMS

Theorem 1.

$$\Delta_5(n) = (-1)^n \sum_{j=0}^{\infty} q_{3,6}^d(n - (3j^2 \pm 2j)),$$

where $q_{3,6}^d(n)$ denotes the number of partitions of n into distinct parts each of which is congruent to 3 (modulo 6), $q_{3,6}^d(0) = 1$, and where the sum extends over all integers j for which the arguments of the partition function are nonnegative.

Proof. The generating function for Δ_i is given by

$$\sum_{n=0}^{\infty} \Delta_i(n)x^n = (1 - x + x^2 - \cdots + (-1)^i x^i)(1 - x^2 + x^4 - \cdots + (-1)^i x^{2i})(1 - x^3 + x^6 + \cdots + (-1)^i x^{3i}) \cdots$$

(1)

$$= \prod_{j=1}^{\infty} (1 - x^j + x^{2j} - \cdots + (-1)^i x^{ij}) = \prod_{j=1}^{\infty} \frac{1 + (-1)^i x^{(i+1)j}}{1 + x^j}.$$

Therefore,

$$(2)\qquad \sum_{n=0}^{\infty} \Delta_5(n)x^n = \prod_{j=1}^{\infty} \frac{1-x^{6j}}{1+x^j} = \prod_{j=1}^{\infty} \frac{(1-x^{6j})(1-x^j)}{1-x^{2j}} = \prod_{j=1}^{\infty} (1-x^{6j})(1-x^{2j-1})$$

$$= \prod_{j=0}^{\infty} (1-x^{6j+1})(1-x^{6j+5})(1-x^{6j+6}) \prod_{j=0}^{\infty} (1-x^{6j+3}).$$

Applying Jacobi's identity

$$(3)\qquad \prod_{j=0}^{\infty} (1-x^{2kj+k-\ell})(1-x^{2kj+k+\ell})(1-x^{2kj+2k}) = \sum_{j=-\infty}^{\infty} (-1)^j x^{kj^2+\ell j}$$

with $k = 3,\ \ell = 2,$ to the triple product in (2), we obtain

$$(4)\qquad \sum_{n=0}^{\infty} \Delta_5(n)x^n = \sum_{j=-\infty}^{\infty} (-1)^j x^{3j^2+2j} \prod_{j=0}^{\infty} (1-x^{6j+3}) .$$

Since

$$\prod_{j=0}^{\infty} (1-x^{6j+3}) = \sum_{k=0}^{\infty} (-1)^k q_{3,6}^d(k)x^k ,$$

we can write (3) as

$$\sum_{n=0}^{\infty} \Delta_5(n)x^n = \left(\sum_{j=0}^{\infty} (-1)^j x^{3j^2\pm 2j}\right)\cdot\left(\sum_{k=0}^{\infty} (-1)^k q_{3,6}^d(k)x^k\right)$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{\infty} (-1)^j(-1)^{n-(3j^2\pm 2j)} q_{3,6}^d(n-(3j^2\pm 2j)) \right\} x^n$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{\infty} (-1)^{n-(3j^2-j\pm 2j)} q_{3,6}^d(n-(3j^2\pm 2j)) \right\} x^n .$$

But $3j^2 - j \pm 2j \equiv 0 \pmod 2$. Hence

$$\sum_{n=0}^{\infty} \Delta_5(n)x^n = \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{\infty} (-1)^n q_{3,6}^d(n-(3j^2\pm 2j)) \right\} x^n .$$

Equating coefficients on both sides, we obtain the theorem.

To illustrate that Theorem 1 allows very rapid calculation of $\Delta_5(n)$, we consider the case $n = 20$, for which we have

$$\Delta_5(20) = \left(\sum_{j=0}^{\infty} q_{3,6}^d(20-(3j^2\pm 2j)) \right) = q_{3,6}^d(15) + q_{3,6}^d(12) = 2 ,$$

all other terms in the sum being 0. This checks with

$$q_5^e(20) - q_5^o(20) = 236 - 234 = 2 ,$$

obtained by computer.

Theorem 2.

$$\Delta_7(n) = (-1)^n \sum_{j=0}^{\infty} q_4^d(n-(2j^2\pm j)) ,$$

where $q_4^d(n)$ denotes the number of partitions of n into distinct parts, each of which is divisible by 4, $q_4^d(0) = 1$, and where the sum extends over all integers j for which the arguments of the partition function are nonnegative.

Proof. Using (1), we have

(5)
$$\sum_{n=0}^{\infty} \Delta_7(n)x^n = \prod_{j=1}^{\infty} \frac{1-x^{8j}}{1+x^j} = \prod_{j=1}^{\infty} \frac{1-x^{4j}}{1+x^j}(1+x^{4j})$$
$$= \prod_{j=0}^{\infty} (1-x^{4j+1})(1-x^{4j+3})(1-x^{4j+4}) \prod_{j=0}^{\infty} (1+x^{4j+4}) .$$

Applying Jacobi's identity (3) with $k = 2$, $\ell = 1$, to the triple product in (5), we obtain

(6)
$$\sum_{n=0}^{\infty} \Delta_7(n)x^n = \sum_{j=-\infty}^{\infty} (-1)^j x^{2j^2+j} \prod_{j=0}^{\infty} (1+x^{4j+4}) = \left(\sum_{j=0}^{\infty} (-1)^j x^{2j^2 \pm j}\right)\left(\sum_{k=0}^{\infty} q_4^d(k)x^k\right)$$
$$= \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{\infty} (-1)^j q_4^d(n-(2j^2 \pm j)) \right\} x^n .$$

Equating coefficients on both sides, we obtain

$$\Delta_7(n) = \sum_{j=0}^{\infty} (-1)^j q_4^d(n-(2j^2 \pm j)) .$$

Now for $n \equiv a \pmod 4$, $0 \leqslant a \leqslant 3$, and observing that $q_4^d(n) = 0$ unless n is divisible by 4, we have

$$\Delta_7(n) = \sum_{\substack{j \leqslant 0 \\ 2j^2 \pm j \equiv a \pmod 4}} (-1)^j q_4^d(n-(2j^2 \pm j))$$
$$= (-1)^a \sum_{\substack{j \geqslant 0 \\ 2j^2 \pm j \equiv a \pmod 4}} q_4^d(n-(2j^2 \pm j)) = (-1)^n \sum_{j=0}^{\infty} q_4^d(n-(2j^2 \pm j)) .$$

The formulae of Theorems 1 and 2 show that $\Delta_i(n)$ for $i = 5$ and 7 is nonnegative if n is even and nonpositive if n is odd.

REFERENCES

1. Ivan Niven and Herbert S. Zuckerman, *An Introduction to the Theory of Numbers,* 3rd ed., John Wiley and Sons, Inc., New York, 1972, pp. 221–222.
2. Dean R. Hickerson, "Identities Relating the Number of Partitions into an Even and Odd Number of Parts," *J. Combinatorial Theory, Section A,* 1973, pp. 351–353.
3. Dean R. Hickerson, oral communication.

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