A GENERAL IDENTITY FOR MULTISECTING GENERATING FUNCTIONS

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Consider the general power series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

(defined for some radius of convergence R, whenever $|x| < R$).

It is desired to find an expression, preferably in terms of $f(x)$, for the so-called multisecting generating function, defined as follows:

$$g(r, s, x) = \sum_{n=0}^{\infty} a_{nr+s} x^{nr+s} \tag{2}$$

(where r and s are integers satisfying $0 \leqslant s < r$).

We shall suppose that $f(x)$, and therefore $g(r,s,x)$ satisfy appropriate convergence requirements, so that the following development may have validity.

The problem indicated above has been solved by various investigators, for certain special cases. For example, Gould [1] has obtained the following results, for the case where $a_n = F_n$ (the n^{th} Fibonacci number):

$$f(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}\,;\; g(r, s, x) = \sum_{n=0}^{\infty} F_{nr+s} x^{nr+s} = \frac{F_s x^s + (-1)^s F_{r-s} x^{r+s}}{1 - L_r x^r + (-1)^r x^{2r}}\,. \tag{3}$$

Also, Hoggatt and Anaya, in a recent joint paper [2], derived a comparable relation for the column generators of Pascal's left-justified triangle. Actually, the definition of the multisecting generating function of $f(x)$ used by these writers was the following:

$$h(r, s, x) = \sum_{n=0}^{\infty} a_{nr+s} x^n\,. \tag{4}$$

The modification of the latter definition given by (2) is slight, since $g(r, s, x)$ and $h(r, s, x)$ are related as follows:

$$g(r, s, x) = x^s h(r, s, x^r)\,. \tag{5}$$

For the purposes of this paper, Eq. (2) is a more convenient definition.

$$g(r, s, x) = \sum_{n=0}^{\infty} a_n \theta(n, r, s) x^n, \quad \text{where} \quad \theta(n, r, s) = \begin{cases} 1 & \text{if } n \equiv s \pmod r \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

This is evident from the definition of $g(r, s, x)$ in (2). Another evident relation is:

$$f(x) = \sum_{s=0}^{r-1} g(r, s, x)\,. \tag{7}$$

What is needed is an explicit expression for $\theta(n,r,s)$. Such an expression is conveniently provided by the following function:

$$\theta(n,r,s) = \frac{1}{r}\sum_{k=0}^{r-1} e^{(n-s)2k\pi i/r} = \frac{e^{(n-s)2k\pi i}-1}{r\left\{e^{(n-s)2k\pi i/r}-1\right\}} \quad (\text{provided } n \not\equiv s \pmod r)\ . \tag{8}$$

If $n = s + mr$, for some integer m, then $e^{(n-s)2k\pi i/r} = e^{2mk\pi i} = 1$. In this event, $\theta(n,r,s) = r/r = 1$. On the other hand, if $n \not\equiv s \pmod r$, the numerator of the second expression in (8) vanishes, but the denominator does not; i.e., $\theta(n,r,s) = 0$. Thus, $\theta(n,r,s)$ as defined in (8) has the desired properties we are seeking for this function. Accordingly,

$$g(r,s,x) = \sum_{n=0}^{\infty} a_n x^n\, \frac{1}{r}\sum_{k=0}^{r-1} e^{(n-s)2k\pi i/r}$$

$$= \frac{1}{r}\sum_{k=0}^{r-1} e^{-2sk\pi i/r} \sum_{n=0}^{\infty} a_n\left\{e^{2k\pi i/r}x\right\}^n = \frac{1}{r}\sum_{k=0}^{r-1} e^{-2sk\pi i/r} f(e^{2k\pi i/r}x)\,.$$

We may make a further simplification, by letting $w(r,k) = e^{2k\pi i/r}$, the $(k+1)^{th}$ r^{th} root of unity. We note that

$$w(r,k) = \left\{w(r,1)\right\}^k ;$$

if we let w_r denote $w(r,1)$, then our relation takes the following form:

$$g(r,s,x) = \frac{1}{r}\sum_{k=0}^{r-1} w_r^{-sk} f(w_r^k x)\ . \tag{9}$$

This is the general expression we are seeking. Any further simplification will depend on the particular values of r and s, and on the specific form of $f(x)$. Indicated below are several special cases of (9) for the first few values of r and s, but for perfectly general $f(x)$:

$$g(1,0,x) = f(x), \quad g(2,0,x) = \tfrac{1}{2}\left\{f(x)+f(-x)\right\}, \quad g(2,1,x) = \tfrac{1}{2}\left\{f(x)-f(-x)\right\},$$

$$g(3,0,x) = \frac{1}{3}\left\{f(x)+f(ux)+f(u^2x)\right\} \ (\text{where } u = \tfrac{1}{2}(-1+i\sqrt{3})), \quad g(3,1,x) = \frac{1}{3}\left\{f(x)+u^2f(ux)+uf(u^2x)\right\},$$

$$g(3,2,x) = \frac{1}{3}\left\{f(x)+uf(ux)+u^2f(u^2x)\right\}, \quad g(4,0,x) = \frac{1}{4}\left\{f(x)+f(ix)+f(-x)+f(-ix)\right\}, \tag{10}$$

$$g(4,1,x) = \frac{1}{4}\left\{f(x)-if(ix)-f(-x)+if(-ix)\right\}, \quad g(4,2,x) = \frac{1}{4}\left\{f(x)-f(ix)+f(-x)-f(-ix)\right\},$$

$$g(4,3,x) = \frac{1}{4}\left\{f(x)+if(ix)-f(-x)-if(-ix)\right\}.$$

Note that the coefficients w_r^{-sk} are themselves r^{th} roots of unity, in permuted order (but with unity itself always first). If we sum $g(r,s,x)$ over s, keeping k fixed, the sum of these coefficients vanishes, except for $k = 0$, where it is unity. This is in accordance with our expected result in (7).

Many interesting special cases of (9) exist, and have been extensively studied, for specific functions $f(x)$. For example, if $f(x) = e^x$, Eq. (9) yields the following:

$$g(r,s,x) = \sum_{n=0}^{\infty} \frac{x^{nr+s}}{(nr+s)!} = \frac{1}{r}\sum_{k=0}^{r-1} w_r^{-sk} e^{w_r^k x}\ . \tag{11}$$

This may be further simplified and expressed as a strictly real function, involving trigonometric terms, but we will not do this here. It will suffice to say that the general form of (9) possesses an intrinsic symmetry which further manipulation tends to eliminate. For example, using identity (11),

$$g(3,0,x) = \frac{1}{3}\left\{e^x + e^{ux} + e^{u^2x}\right\},$$

where u is as defined in (10); however, we may also express $g(3,0,x)$ in real form:

$$g(3,0,x) = \frac{1}{3}\left\{e^x + 2e^{-\frac{1}{2}x}\cos(\tfrac{1}{2}x\sqrt{3})\right\},$$

which is not as elegant a result as (11). Similarly, many special cases of (9) may be verified by the interested reader; it is the writer's opinion, nevertheless, that (9) possesses a special elegance just as it stands, limited though its practical usefulness may be.

REFERENCES

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2. V. E. Hoggatt, Jr., and Janet Crump Anaya, "A Primer for the Fibonacci Numbers—Part XI: Multisection Generating Functions for the Columns of Pascal's Triangle," *The Fibonacci Quarterly*, Vol. 11, No. 1, Feb. 1973, pp. 85–90, 104.

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A FORMULA FOR $A_n^2(x)$

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This paper is a follow-up of [1], which dealt with certain combinatorial coefficients denoted by the symbol $A_n(x)$. We begin by recalling the definition of $A_n(x)$, which was given in [1]:

$$(1)\qquad (1-u)^{-1}(1+u)^x = \sum_{n=0}^{\infty} A_n(x)u^n\,; \quad \text{therefore,} \quad A_n(x) = \sum_{i=0}^{n}\binom{x}{i},$$

which is a polynomial in x. In [1], the writer indicated that he had found the first few terms in the combinatorial expansion for $A_n^2(x)$, but was unable to obtain the general expansion. Formula (78) in [1] gave the first few terms of the expression, derived by direct expansion:

$$(2)\qquad A_n^2(x) = \binom{2n}{n}\left\{\binom{x}{2n} + \tfrac{1}{2}(n+2)\binom{x}{2n-1} + \left(\frac{n^3+2n^2+3n-4}{8n-4}\right)\binom{x}{2n-2} + \cdots\right\}.$$

The problem of obtaining the general term of the polynomial $A_n^2(x)$ has now been resolved. However, the expression is in the form of an iterated summation, which is indicated below:

$$(3)\qquad A_n^2(x) = \sum_{i=0}^{n} 3^i\binom{x}{i} + \sum_{i=n+1}^{2n}\binom{x}{i}\sum_{j=i-n}^{n}\binom{i}{j}\sum_{k=0}^{j+n-i}\binom{j}{k} \qquad (n = 1, 2, 3, \cdots)$$

Perhaps some interested reader can reduce this expression to a simpler one, involving only two (or possibly one) summation variables. If we denote the coefficient of $\binom{x}{i}$ as θ_i, relation (3) above yields the following values:

$$\theta_{2n} = \binom{2n}{n}; \quad \theta_{2n-1} = \frac{(2n-1)!}{n!n!}\,n(n+2); \quad \theta_{2n-2} = \frac{(2n-2)!}{n!(n-1)!}\,\tfrac{1}{2}(n^3+2n^2+3n-4)$$

(these last three values may be compared with those in (2));

$$\theta_{2n-3} = \frac{(2n-3)}{n!(n-2)!}\,\frac{1}{6}(n^4+n^3+8n^2+2n-24);$$

also, $\theta_{n+1} = 3^{n+1} - 2\cdot 2^{n+1} + 1^{n+1}$; $\quad \theta_{n+2} = 3^{n+2} - 2\cdot 2^{n+2} + 1^{n+2} - (n+2)(2^{n+2}-1) + (n+2)^2$.

In attempting to discover the law of formation of θ_i for $i > n$, it is clear that increasing difficulty is encountered as one recedes from either end of the second (iterated) summation in the right member of (3). Possibly, θ_i may be concisely expressed in terms of a finite difference operator, but this approach has not yet been fully explored.

A proof of (3) follows. The proof hinges on a formula due to Riordan, indicated as formula (6.44) in [2]. This formula is as follows:

$$(4)\qquad \sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{m-k}\binom{x}{m+n-k} = \binom{x}{m}\binom{x}{n}.$$

A slightly more convenient form of (4) is obtained by the substitution $i = m+n-k$, also observing that the upper limit in (4) need only equal min (m,n), since subsequent terms vanish. Then (4) takes the following form:

$$(5)\qquad \binom{x}{m}\binom{x}{n} = \sum_{i=\max(m,n)}^{m+n}\binom{x}{i}\binom{i}{m}\binom{m}{i-n} = \sum_{i=\max(m,n)}^{m+n}\binom{x}{i}\binom{i}{n}\binom{n}{i-m}.$$