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A FORMULA FOR $A_n^2(x)$

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This paper is a follow-up of [1], which dealt with certain combinatorial coefficients denoted by the symbol $A_n(x)$. We begin by recalling the definition of $A_n(x)$, which was given in [1]:

$$(1)\qquad (1-u)^{-1}(1+u)^x = \sum_{n=0}^{\infty} A_n(x)u^n\,; \quad \text{therefore,} \quad A_n(x) = \sum_{i=0}^{n}\binom{x}{i}\,,$$

which is a polynomial in x. In [1], the writer indicated that he had found the first few terms in the combinatorial expansion for $A_n^2(x)$, but was unable to obtain the general expansion. Formula (78) in [1] gave the first few terms of the expression, derived by direct expansion:

$$(2)\qquad A_n^2(x) = \binom{2n}{n}\left\{\binom{x}{2n} + \tfrac{1}{2}(n+2)\binom{x}{2n-1} + \left(\frac{n^3+2n^2+3n-4}{8n-4}\right)\binom{x}{2n-2} + \cdots\right\}\,.$$

The problem of obtaining the general term of the polynomial $A_n^2(x)$ has now been resolved. However, the expression is in the form of an iterated summation, which is indicated below:

$$(3)\qquad A_n^2(x) = \sum_{i=0}^{n} 3^i\binom{x}{i} + \sum_{i=n+1}^{2n}\binom{x}{i}\sum_{j=i-n}^{n}\binom{i}{j}\sum_{k=0}^{j+n-i}\binom{j}{k} \qquad (n = 1, 2, 3, \cdots)$$

Perhaps some interested reader can reduce this expression to a simpler one, involving only two (or possibly one) summation variables. If we denote the coefficient of $\binom{x}{i}$ as θ_i, relation (3) above yields the following values:

$$\theta_{2n} = \binom{2n}{n}\,; \quad \theta_{2n-1} = \frac{(2n-1)!}{n!n!}\,n(n+2); \quad \theta_{2n-2} = \frac{(2n-2)!}{n!(n-1)!}\,\tfrac{1}{2}(n^3+2n^2+3n-4)$$

(these last three values may be compared with those in (2));

$$\theta_{2n-3} = \frac{(2n-3)}{n!(n-2)!}\,\frac{1}{6}(n^4+n^3+8n^2+2n-24);$$

also, $\theta_{n+1} = 3^{n+1} - 2\cdot 2^{n+1} + 1^{n+1}$; $\quad \theta_{n+2} = 3^{n+2} - 2\cdot 2^{n+2} + 1^{n+2} - (n+2)(2^{n+2}-1) + (n+2)^2$.

In attempting to discover the law of formation of θ_i for $i > n$, it is clear that increasing difficulty is encountered as one recedes from either end of the second (iterated) summation in the right member of (3). Possibly, θ_i may be concisely expressed in terms of a finite difference operator, but this approach has not yet been fully explored.

A proof of (3) follows. The proof hinges on a formula due to Riordan, indicated as formula (6.44) in [2]. This formula is as follows:

$$(4)\qquad \sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{m-k}\binom{x}{m+n-k} = \binom{x}{m}\binom{x}{n}\,.$$

A slightly more convenient form of (4) is obtained by the substitution $i = m+n-k$, also observing that the upper limit in (4) need only equal min (m,n), since subsequent terms vanish. Then (4) takes the following form:

$$(5)\qquad \binom{x}{m}\binom{x}{n} = \sum_{i=\max(m,n)}^{m+n}\binom{x}{i}\binom{i}{m}\binom{m}{i-n} = \sum_{i=\max(m,n)}^{m+n}\binom{x}{i}\binom{i}{n}\binom{n}{i-m}\,.$$

Now

$$A_n^2(x) = \sum_{j=0}^{n} \binom{x}{j} \sum_{h=0}^{n} \binom{x}{h} = \sum_{j=0}^{n} \sum_{h=0}^{n} \sum_{i=\max(j,h)}^{j+h} \binom{x}{i}\binom{i}{j}\binom{j}{i-h},$$

(applying the result in (5)),

$$= \sum_{j=0}^{n} \sum_{i=j}^{j+n} \binom{x}{i}\binom{i}{j} \sum_{h=i-j}^{m} \binom{j}{i-h},$$

where $m = \min(i,n)$. Now let $h = i - j + k$. Then

$$A_n^2(x) = \sum_{j=0}^{n}\sum_{i=j}^{j+n} \binom{x}{i}\binom{i}{j} \sum_{k=0}^{m-i+j} \binom{j}{j-k} = \sum_{i=0}^{2n} \binom{x}{i} \sum_{j=i-m}^{m} \binom{i}{j} \sum_{k=0}^{m-i+j} \binom{j}{k}.$$

Distinguishing between the cases where $i \leqslant n$ and $i > n$, this expression may be simplified as follows:

$$\sum_{i=0}^{n} \binom{x}{i} \sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} + \sum_{i=n+1}^{2n} \binom{x}{i} \sum_{j=i-n}^{n} \binom{i}{j} \sum_{k=0}^{n-i+j} \binom{j}{k}.$$

Comparing this with the right member of (3), we see that the only thing left to prove is that

$$3^i = \sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k}.$$

But this is an easy consequence of the binomial theorem, applied twice, since

$$\sum_{k=0}^{j} \binom{j}{k} = (1+1)^j = 2^j, \quad \text{and} \quad \sum_{j=0}^{i} \binom{i}{j} 2^j = (1+2)^i = 3^i.$$

Hence (3) is proved. Obviously, the expression for θ_i given by (3), for $i > n$, is not unique. By various substitutions and/or translations, a wide variety of expressions for θ_i may be derived from the basic relationship in (3). For example, the following alternative formula is given, without proof:

$$\text{(6)} \qquad \sum_{j=[\frac{1}{2}(1+i)]}^{n} \binom{i}{j} \sum_{k=0}^{2j-i} \binom{1+j}{k} = \sum_{j=2i-2n}^{i} \binom{i}{j} \sum_{k=i-n}^{j+n-i} \binom{j}{k} = \theta_i, \quad (i > n)$$

(where $[u]$ represents the integral part of u).

Attempts by the writer to obtain a generating function for the $A_n^2(x)$'s, in closed form, were unsuccessful. Can anyone help?

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