RELATIONS BETWEEN EULER AND LUCAS NUMBERS

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1. INTRODUCTION

In a previous article [1], the author presented a class of relations between Fibonacci-Lucas sequences and the generalized number sequences of Bernoulli. The same simple techniques can be used to obtain such identities involving other classical numbers.

The purpose of the present paper is to give explicit new relations and identities that involve *Lucas* numbers together with the famous numbers of *Euler.*

2. PRELIMINARIES

EULER NUMBERS

The *generalized Euler numbers* $E_n^{(m)}$ of the m^{th} *order* are defined by the generating function (see, for example [3]),

$$\frac{2^m}{(e^t+e^{-t})^m} = (\operatorname{sech} t)^m = \sum_{n=0}^{\infty} E_n^{(m)} \frac{t^n}{n!}, \qquad |t| < \pi/2\,. \tag{1}$$

If $m = 1$, one writes $E_n^{(1)} \equiv E_n$, and has the more familiar Euler number sequence of the *first order:* 1, 0, –1, 0, 5, 0, –61, 0, 1385, 0, –50521, ⋯. The generalized numbers satisfy the *partial difference equation*

$$mE_n^{(m+1)} - E_{n+1}^{(m)} - mE_n^{(m)} = 0. \tag{2}$$

Moreover,

$$E_n^{(m)} = \frac{d^n}{dt^n}\,[(\operatorname{sech} t)^m]_{t=0}\,, \tag{3}$$

so one obtains the sequence

$$E_0^{(m)} = 1, \qquad E_1^{(m)} = 0, \qquad E_2^{(m)} = -m, \qquad E_3^{(m)} = 0, \qquad E_4^{(m)} = m(3m+2), \cdots, \tag{4}$$

with $E_{2k-1}^{(m)} = 0$ for $k \geqslant 1$.

If m is a *negative* integer, i.e., when $m = -p,\ p \geqslant 1$, the relation

$$E_n^{(-p)} = \frac{d^n}{dt^n}\,[(\cosh t)^p]_{t=0} \tag{5}$$

yields the explicit formula

$$E_{2k}^{(-p)} = \frac{1}{2^p}\sum_{j=0}^{p}\binom{p}{j}(p-2j)^{2k}, \qquad k \geqslant 0\,. \tag{6}$$

Euler and Bernoulli numbers of the first kind ($m = 1$) are related by the two equations

$$B_{2k} = \frac{-2k}{4^k(4^k-1)}\sum_{j=0}^{k-1}\binom{2k-1}{2j}E_{2j}, \qquad E_{2n} = 2 - \frac{1}{2n+1}\sum_{k=0}^{n}\binom{2n+1}{2k}(16)^k B_{2k}\,. \tag{7}$$

(See [2].)

LUCAS AND FIBONACCI NUMBERS

If

$$a = (1+\sqrt{5})/2 \quad \text{and} \quad b = (1-\sqrt{5})/2, \tag{8}$$

then the *Fibonacci* and *Lucas numbers* are defined respectively by the generating formulas

(9) $$\frac{e^{at}-e^{bt}}{\sqrt{5}} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}, \qquad e^{at}+e^{bt} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!},$$

or explicitly by the equations

(10) $$F_n = \frac{a^n-b^n}{a-b}, \qquad L = a^n+b^n, \qquad n \geqslant 0$$

3. SOME IDENTITIES

With the above background preliminaries, we are in immediate position to obtain three identities. As in the previous article [1], we shall use inventive series manipulation as the fundamental method.

EXAMPLE 1

Note that

(11) $$e^{at}+e^{bt} = e^{t/2}(e^{ct}+e^{-ct}) = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!},$$

where the quantity c, which will occur frequently in subsequent equations, is

(12) $$c = \sqrt{5}/2.$$

We also have

$$\frac{e^{t/2}}{e^{at}+e^{bt}} = \frac{1}{e^{ct}+e^{-ct}} = \tfrac{1}{2}\sum_{n=0}^{\infty} c^n E_n \frac{t^n}{n!}$$

or

(13) $$e^{at}+e^{bt} = \frac{2e^{t/2}}{\sum\limits_{n=0}^{\infty} c^n E_n \frac{t^n}{n!}},$$

where we have made use of Eq. (1) with m = 1. Thus,

(14) $$\left[\sum_{n=0}^{\infty} c^n E_n \frac{t^n}{n!}\right]\left[\sum_{s=0}^{\infty} L_s \frac{t^s}{s!}\right] = 2e^{t/2} = \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} \frac{t^n}{n!}.$$

Application of Cauchy's rule for multiplying power series now yields

(15) $$\sum_{k=0}^{n} \binom{n}{k} c^k E_k L_{n-k} = 2^{1-n} \qquad n \geqslant 0.$$

Since $E_{2m-1} = 0$ for $m \geqslant 1$, and since $c = \sqrt{5}/2$, we have the identity*

(16) $$\sum_{k=0}^{[n/2]} \binom{n}{2k}\left(\frac{5}{4}\right)^k E_{2k} L_{n-2k} = 2^{1-n}$$

involving Euler numbers of the *first order* and the Lucas numbers. This identity holds for all $n \geqslant 0$.

EXAMPLE 2

Now

$$(e^{at}+e^{bt})^2 = e^t(e^{ct}+e^{-ct})^2,$$

or, in view of Eq. (1),

(17) $$\frac{e^t}{(e^{at}+e^{bt})^2} = \frac{1}{(e^{ct}+e^{-ct})^2} = \tfrac{1}{4}\sum_{n=0}^{\infty} c^n E_n^{(2)} \frac{t^n}{n!},$$

*This particular identity is also found in [4].

where $E_n^{(2)}$ are Euler numbers of the *second order.* But it is also seen, using the second generating function in relations (9), that

$$(18)\qquad (e^{at}+e^{bt})^2 = [e^{2at}+e^{2bt}]+2e^t = \sum_{n=0}^{\infty} [2^n L_n + 2]\,\frac{t^n}{n!} \quad .$$

So, with (17) and (18), one has

$$\left[\sum_{n=0}^{\infty} c^n E_n^{(2)} \frac{t^n}{n!}\right]\left[\sum_{s=0}^{\infty} (2^s L_s + 2)\frac{t^s}{s!}\right] = 4e^t = 4\sum_{n=0}^{\infty}\frac{t^n}{n!},$$

or the identity

$$\sum_{k=0}^{\infty} \binom{n}{k} c^k E_k^{(2)} [2^{n-k} L_{n-k} + 2] = 4\,.$$

Since the odd Euler numbers are zero, this can be written as

$$(19)\qquad \sum_{n=0}^{[n/2]} \binom{n}{2k}\left(\frac{5}{4}\right)^k E_{2k}^{(2)}\, [2^{n-2k} L_{n-2k} + 2] = 4, \qquad n \geq 0.$$

EXAMPLE 3

Again, we have

$$(20)\qquad (e^{at}+e^{bt})^2 = 4e\,\frac{(e^{ct}+e^{-ct})^2}{4}$$

$$= 4\left[\sum_{s=0}^{\infty}\frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty} c^n E_n^{(-2)}\frac{t^n}{n!}\right],$$

where $E_n^{(-2)}$ are Euler numbers of *negative* second order. Once more we note that

$$(21)\qquad (e^{at}+e^{bt})^2 = \sum_{n=0}^{\infty} [2^n L_n + 2]\,\frac{t^n}{n!},$$

and then equate this to the expression on the right in (20). Thus

$$[2^n L_n + 2] = 4\sum_{k=0}^{n}\binom{n}{k} c^k E_k^{(-2)},$$

furnishing the identity

$$(22)\qquad L_n = 2^{1-n}\left[-1+2\sum_{k=0}^{[n/2]}\binom{n}{2k}\left(\frac{5}{4}\right)^k E_{2k}^{(-2)}\right], \qquad n \geq 0.$$

4. GENERALIZATION

The procedure just illustrated can easily be extended to furnish a whole new class of similar identities involving Lucas numbers and Euler numbers of higher order.

GENERAL CASE WHEN m IS AN ARBITRARY NEGATIVE INTEGER

We take $m = -p$, with p being a positive integer ≥ 1. From Eq. (1) it is seen that

$$(23)\qquad \frac{(e^t+e^{-t})^p}{2} = (\cosh t)^p = \sum_{n=0}^{\infty} E_n^{(-p)}\frac{t^n}{n!},$$

where $E_n^{(-p)}$ are Euler numbers of the $(-p)^{th}$ order. We also note that

$$(24)\quad (e^{at}+e^{bt}) = [e^{pat}+e^{pbt}] + \sum_{r=1}^{p-1} \binom{p}{r} e^{[pa+(b-a)r]t} = \sum_{n=0}^{\infty} \left\{ p^n L_n + \sum_{r=1}^{p-1} \binom{p}{r} [pa+(b-a)r]^n \right\} \frac{t^n}{n!},$$

and that

$$(25)\quad (e^{at}+e^{bt})^p = 2^p e^{pt/2} \frac{(e^{ct}+e^{-ct})^p}{2^p} = 2^p \left[\sum_{s=0}^{\infty} \left(\frac{p}{2}\right)^s \frac{t^s}{s!} \right] \left[\sum_{n=0}^{\infty} c^n E_n^{(-p)} \frac{t^n}{n!} \right].$$

Equating (24) and (25) now yields

$$(26)\quad p^n L_n + \sum_{r=1}^{p-1} \binom{p}{r} [pa+(b-a)r]^n = 2^p \sum_{k=0}^{n} \binom{n}{k} \left(\frac{p}{2}\right)^{n-k} c^k E_k^{(-p)},$$

or the identity

$$(27)\quad L_n = p^{-n} \left\{ -\sum_{r=1}^{p-1} \binom{p}{r} [pa+(b-a)r]^n + 2^p \sum_{k=0}^{[n/2]} \binom{n}{2k} \left(\frac{p}{2}\right)^{n-2k} \left(\frac{5}{4}\right)^k E_{2k}^{(-p)} \right\}.$$

This identity holds for each $p \geqslant 2$, and it furnishes an infinite number of identities. In the special case when $p = 1$, we have

$$(28)\quad L_n = 2^{1-n} \sum_{k=0}^{[n/2]} \binom{n}{2k} 5^k E_{2k}^{(-1)}, \qquad n \geqslant 0.$$

Equation (27) is remarkable in that it embodies explicit formulas for expressing any Lucas number in a finite sum involving any particular Euler sequence of negative order that one may choose.

GENERAL CASE WHEN m IS A POSITIVE INTEGER

Different types of identities are obtained when m is positive, but the technique of deriving them is the same. We present the result without showing the detailed development. It is as follows:

$$(29)\quad \sum_{k=0}^{[n/2]} \binom{n}{2k} \left(\frac{5}{4}\right)^k E_{2k}^{(m)} \left\{ m^{n-2k} L_{n-2k} + \sum_{r=1}^{m-1} \binom{m}{r} [ma+(b-a)r]^n \right\} = 2^{m-n} m^n$$

which reduces to (16) when $m = 1$, and to (19) when $m = 2$. The identity (29) holds for all positive m, and represents a one-parameter family of identities that are valid for all $n \geqslant 0$.

5. REMARKS

By using the first equation given in (7), other identities, involving Fibonacci numbers and Euler numbers, can be found if B_{2k} in terms of Euler numbers is inserted in the identities obtained in [1].

Since

$$(30)\quad F_n = \frac{1}{5} [L_{n+1} + L_{n-1}], \qquad n \geqslant 1$$

Equation (27) can easily be used to explicitly express any Fibonacci number in terms that involve any Euler sequence of negative order.

It may interest the reader to extend our identities and to investigate how such relations may be applied. The author (as every Fibonacci-number enthusiast should do after recording his formulas) is turning his attention to the question of what might be *done* with them.

REFERENCES

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