SOME IDENTITIES OF BRUCKMAN

L. CARLITZ*
Duke University, Durham, North Carolina 27706

!. Bruckman [1] defined a sequence of numbers $\{A_n\}$ by means of

$$(1.1)\qquad (1-z)^{-1}(1+z)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} A_n z^n ,$$

so that

$$(1.2)\qquad A_n = \sum_{k=0}^{n} (-1)^k 2^{-2k} \binom{2k}{k} .$$

He proved the striking result

$$(1.3)\qquad \sum_{n=0}^{\infty} 2^{2n} \frac{n!n!}{(2n+1)!} A_n^2 x^{2n+1} = \frac{\arctan x}{\sqrt{1-x^2}} ,$$

which is equivalent to

$$(1.4)\qquad A_n^2 = 2^{-2n} \binom{2n}{n} \sum_{k=0}^{n} (-1)^{n-k} 2^{-2k} \binom{2k}{k} \frac{2n+1}{2n-k+1} .$$

Gould [4] has discussed Bruckman's results in some detail and indicated their relationship to earlier results. He remarks that "a direct proof of (1.4) by squaring (1.2) is by no means trivial." However, he does not give a proof of the formula.

The purpose of this note is to show that (1.3) is a very special case of a much more general result involving hypergeometric polynomials. We also show how a generalized version of (1.3) can be obtained using a little calculus.

2. In the standard notation put

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!\,(c)_n} z^n ,$$

where

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1.$$

Weisner [6] has proved the formula

$$(2.1)\qquad \sum_{n=0}^{\infty} \frac{(c)_n z^n}{n!} F_n(-n,a;c;x)F(-n,b;c;y)$$
$$= (1-z)^{a+b-c}(1+(x-1)z)^{-a}(1+(y-1)z)^{-b} F(a,b;c;\zeta),$$

where

$$(2.2)\qquad \zeta = \frac{xyz}{(1+(x-1)z)(1+(y-1)z)} .$$

This result had indeed been proved earlier by Meixner [5]. For an elementary proof of (2.1) see [3].

Replacing x, y by $1-x$, $1-y$, respectively, Eq. (2.1) becomes

*Supported in part by NSF Grant GP-37924

(2.3)
$$\sum_{n=0}^{\infty} \frac{(c)_n z^n}{n!} F_n(-n, a; c; 1-x) F_n(-n, b; c; 1-y)$$
$$= (1-z)^{a+b-c}(1-xz)^{-a}(1-yz)^{-b}F(a,b;c;\xi),$$

where

(2.4)
$$\xi = \frac{(1-x)(1-y)z}{(1-xz)(1-yz)}.$$

In particular, for $c = a + b$, Eq. (2.3) reduces to

(2.5)
$$\sum_{n=0}^{\infty} \frac{(a+b)_n z^n}{n!} F_n(-n,a;a+b;1-x)F(-n,b;a+b;1-y)$$
$$= (1-xz)^{-a}(1-yz)^{-b}F(a,b;c;\xi).$$

Consider

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n!} F(-n,a;c;1-x)z^n = \sum_{n=0}^{\infty} \frac{(c)_n z^n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (a)_k}{k!(c)_k} (1-x)^k$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!} (1-x)^k z^k \sum_{n=k}^{\infty} \frac{(c+k)_{n-k}}{(n-k)!} z^{n-k}$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k}{k!} (1-x)^k z^k (1-z)^{-c-k} = (1-z)^{-c}\left(1 + \frac{(1-x)z}{1-z}\right)^{-a},$$

where we have used

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}.$$

It follows that

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n!} F(-n,a;c;1-x)z^n = (1-z)^{a-c}(1-xz)^{-a}.$$

Thus, for $c = a + b$, we have

(2.6)
$$\sum_{n=0}^{\infty} \frac{(a+b)_n}{n!} F(-n,a;a+b;1-x) = (1-z)^{-b}(1-xz)^{-a}.$$

It follows that

(2.7)
$$F(-n,a;a+b;1-x) = x^n F(-n,b;a+b;1-x^{-1}).$$

3. We now specialize (2.5) by taking

(3.1)
$$a = \tfrac{1}{2}, \qquad b = 1, \qquad c = 3/2.$$

Then (2.5) becomes

(3.2)
$$\sum_{n=0}^{\infty} \frac{(3/2)_n z^n}{n!} F_n(-n,\tfrac{1}{2};3/2;1-x)F(-n,1;3/2;1-y) = (1-xz)^{-\frac{1}{2}}(1-yz)^{-1}F(\tfrac{1}{2},1;3/2;\quad).$$

In view of (2.7) this may be replaced by

(3.3)
$$\sum_{n=0}^{\infty} \frac{(3/2)_n y^n z^n}{n!} F_n(-n,\tfrac{1}{2};3/2;1-x)F(-n,\tfrac{1}{2};3/2;1-y^{-1}) = (1-xz)^{-\frac{1}{2}}(1-yz)^{-1}F(\tfrac{1}{2},1;3/2;\quad).$$

We define the polynomial $A_n(x)$ by means of

(3.4) $$\sum_{n=0}^{\infty} A_n(x)z^n = (1-z)^{-1}(1-xz)^{-1/2} .$$

This is equivalent to

(3.5) $$A_n(x) = \sum_{k=0}^{n} 2^{-2k}\binom{2k}{k}x^k .$$

Comparing (3.4) with (1.1) or (3.5) with (1.2), it is evident that

(3.6) $$A_n = A_n(-1) .$$

It will also be convenient to define

(3.7) $$\bar{A}_n(x) = x^n A_n(x^{-1}) = \sum_{k=0}^{n} 2^{-2k}\binom{2k}{k}x^{n-k} .$$

Comparing (3.4) with (2.6), we get

(3.8) $$A_n(x) = \frac{(3/n)_n}{n!} F(-n, 1/2; 3/2; 1-x).$$

Thus (3.3) becomes

(3.9) $$\sum_{n=0}^{\infty} \frac{n!z^n}{(3/2)_n} A_n(x)\bar{A}_n(y) = (1-xz)^{-1/2}(1-yz)^{-1}F(1/2,1;3/2;\quad).$$

Since

$$(3/2)_n = 2^{-2n}\frac{n!}{(2n+1)!}$$

and

$$zF(1/2,1;3/2;-z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} = \arctan z,$$

(3.9) may be replaced by

(3.10) $$\sum_{n=0}^{\infty} (-1)^n 2^{2n}\frac{n!n!}{(2n+1)!} z^{2n+1}A_n(x)\bar{A}_n(y)$$

$$= \left\{(1-x)(1-y)(1+yz^2)\right\}^{-1/2}\arctan\left\{z\left(\frac{(1-x)(1-y)}{(1+xz^2)(1+yz^2)}\right)\right\}^{1/2} .$$

For $x=y=-1$, it is evident from (3.6) and (3.7) that

(3.11) $$\sum_{n=0}^{\infty} 2^{2n}\frac{n!n!}{(2n+1)!}A_n^2z^{2n+1} = 1/2(1-z^2)^{-1/2}\arctan\frac{2z}{1-z^2} .$$

For $y=x$, the right-hand side of (3.10) becomes

$$(1-x)^{-1}(1+xz^2)^{-1/2}\arctan\frac{(1-x)z}{1+xz^2} = \sum_{k=0}^{\infty}(-1)^k\frac{(1-x)^{2k}z^{2k+1}}{2k+1}(1+xz^2)^{-2k-(3/2)}$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{(1-x)^{2k}z^{2k+1}}{2k+1}\sum_{j=0}^{\infty}(-1)^j\binom{2k+j+1/2}{j}x^jz^{2j} .$$

Comparing coefficients of z^{2n+1} we get

(3.12) $$2^{2n}A_n(x)\bar{A}_n(x) = \frac{(2n+1)!}{n!n!}\sum_{j=0}^{n}\binom{2n-j+1/2}{j}\frac{x^j(1-x)^{2n-2j}}{2n-2j+1} .$$

The corresponding formula for $A_n(x)\bar{A}_n(y)$ is more complicated and will be omitted.

For $x = -1$, (3.12) reduces to

$$A_n^2 = \frac{(2n+1)!}{n!n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{2n-j+\frac{1}{2}}{j}\frac{2^{-2j}}{2n-2j+1}, \tag{3.13}$$

which may be compared with (1.4).

Formulas (3.11) and (1.3) are equivalent. This is a consequence of

$$\arctan\frac{2z}{1-z^2} = 2\arctan z.$$

We remark that in a recent paper [2] Bruckman has considered a different generalization of A_n.

4. We can also prove (3.10) in the following way. To begin with, take

$$(1-z)^{-1}(1-xz)^{-\frac{1}{2}} = (1-z)^{-(3/2)}\left(1+\frac{(1-x)z}{1-z}\right)^{\frac{1}{2}} = \sum_{k=0}^{\infty}(-1)^k 2^{-2k}\binom{2k}{k}\frac{(1-x)^k z^k}{(1-z)^{k+(3/2)}}$$

$$= \sum_{k=0}^{\infty}(-1)^k 2^{-2k}\binom{2k}{k}(1-x)^k z^k\sum_{j=0}^{\infty}\binom{k+j+\frac{1}{2}}{j}z^j.$$

It then follows from (3.4) that

$$A_n(x) = \sum_{k=0}^{n}(-1)^k 2^{-2k}\binom{2k}{k}\binom{n+\frac{1}{2}}{n-k}(1-x)^k = 2^{-2n}\frac{(2n+1)!}{n!n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1-x)^k}{2k+1}. \tag{4.1}$$

Since

$$\int_0^1 (1-(1-x)t^2)dt = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1-x)^k}{2k+1},$$

it follows that

$$A_n(x) = 2^{-2n}\frac{(2n+1)!}{n!\,n!}\int_0^1 (1-(1-x)t^2)^n dt \tag{4.2}$$

and

$$\bar{A}_n(x) = 2^{-2n}\frac{(2n+1)!}{n!n!}\int_0^1 (x+(1-x)t^2)^n dt. \tag{4.3}$$

Thus

$$\sum_{n=0}^{\infty}(-1)^n 2^{2n}\frac{n!n!}{(2n+1)!}A_n(x)\bar{A}_n(y)z^{2n+1} = \sum_{n=0}^{\infty}(-1)^n A_n(x)z^{2n+1}\int_0^1 (y+(1-y)t^2)^n dt$$

$$= z\int_0^1 \left\{1+(y+(1-y)t^2)z^2\right\}^{-1}\left\{1+x(y+(1-y)t^2)z^2\right\}^{-\frac{1}{2}}dt.$$

We shall make use of the formula

$$\int\frac{dt}{(a'+b't^2)(a+bt^2)^{\frac{1}{2}}} = \frac{1}{(a'(ab'-a'b))^{\frac{1}{2}}}\arctan\left\{x\left(\frac{ab'-a'b}{a'(a+bx^2)}\right)\right\}^{\frac{1}{2}} \tag{4.4}$$

where

$$\begin{cases} a = 1+xyz^2, & b = x(1-y)z^2 \\ a' = 1+yz^2, & b' = (1-y)z^2 \end{cases},$$

$$ab' = a'b = (1-x)(1-y)z^2, \qquad a+b = 1+xz^2.$$

We therefore get

$$\sum_{n=0}^{\infty} (-1)^n 2^{2n} \frac{n!n!}{(2n+1)!} A_n(x)\bar{A}_n(y)z^{2n+1}$$

$$= \left\{(1-x)(1-y)(1+xz^2)\right\}^{-\frac{1}{2}} \arctan \left\{ z \left(\frac{(1-x)(1-y)}{(1+xz^2)(1+yz^2)} \right) \right\}^{\frac{1}{2}}$$

which is identical with (3.10).

APPENDIX

5. We shall prove the following identity:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} c_r x^r \sum_{s=0}^{n} (-1)^s \binom{n}{s} d_s y^s \tag{5.1}$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(xyz)^n}{(1-z)^{2n}} C_n(\lambda, x, z) D_n(\lambda, y, z) ,$$

where $\{c_n\}$, $\{d_n\}$ are sequences of arbitrary complex numbers and

$$C_n(\lambda, x, z) = \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} c_{n+r} \left(\frac{-xz}{1-z} \right)^r, \qquad D_n(\lambda, y, z) = \sum_{s=0}^{\infty} \frac{(\lambda+n)_s}{s!} d_{n+s} \left(\frac{-yz}{1-z} \right)^s .$$

We may think of (4.1) as an identity between formal power series.

PROOF OF (5.1). The left-hand side of (4.1) is equal to

$$\sum_{r,s=0}^{\infty} (-1)^{r+s} c_r d_s x^r y^s \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \binom{n}{r}\binom{n}{s} z^n . \tag{5.2}$$

The right-hand side of (5.1) is equal to

$$(1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (xyz)^n \sum_{r=0}^{\infty} \frac{(\lambda+n)_r}{r!} c_{n+r}(-xz)^r \sum_{s=0}^{\infty} \frac{(\lambda+n)_s}{s!} d_{n+s}(-yz)^s (1-z)^{-2n-r-s}$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \sum_{r,s=0}^{\infty} (-1)^{r+s} \frac{(\lambda)_{n+r}(\lambda)_{n+s}}{n!\,r!\,s!\,(\lambda)_n} c_{n+r} d_{n+s} x^{n+r} y^{n+s} z^{n+r+s} (1-z)^{-2n-r-s}$$

$$= (1-z)^{-\lambda} \sum_{r,s=0}^{\infty} (-1)^{r+s} (\lambda)_r (\lambda)_s c_r d_s x^r y^s (1-z)^{-r-s} \sum_{n=0}^{\min(r,s)} \frac{z^{r+s-n}}{n!(r-n)!(s-n)!(\lambda)_n} .$$

Comparing this with (5.2), it is evident that it suffices to show that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \binom{n}{r}\binom{n}{s} z^n = (\lambda)_r(\lambda)_s (1-z)^{-\lambda-r-s} \sum_{n=0}^{\min(r,s)} \frac{z^{r+s-n}}{n!(r-n)!(s-n)!(\lambda)_n} . \tag{5.3}$$

If we multiply both sides of (5.3) by $x^r y^s$, and sum over r, s, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1+x)^n (1+y)^n z^n = (1-z)^{-\lambda} \sum_{r,s=0}^{\infty} (\lambda)_r (\lambda)_s \frac{x^r y^s}{(1-z)^{r+s}} \sum_{n=0}^{\min(r,s)} \frac{z^{r+s-n}}{n!(r-n)!(s-n)!(\lambda)_n}$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{(xyz)^n}{(1-z)^{2n}} \sum_{r,s=0}^{\infty} \frac{(\lambda+n)_r(\lambda+n)_s}{r!s!} \frac{(xz)^r(yz)^s}{(1-z)^{r+s}} = (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}$$

$$\cdot \frac{(xyz)^n}{(1-z)^{2n}} \left(1 - \frac{xz}{1-z} \right)^{-\lambda-n} \left(1 - \frac{yz}{1-z} \right)^{-\lambda-n} = (1-z)^{\lambda}(1-(1+x)z)^{-\lambda}(1-(1+y)z)^{-\lambda}$$

$$\cdot \left[1 - \frac{xyz}{(1-(1+x)z)(1-(1+y)z)} \right]^{-\lambda} = (1-z)^{\lambda} \left\{ [1-(1+x)z]\,[1-(1+y)z] - xyz \right\}^{-\lambda} .$$

Thus (5.3) is equivalent to

$$[1-(1+x)(1+y)z]^{-\lambda} = (1-z)^{\lambda}\left\{[1-(1+x)z][1-(1+y)z]-xyz\right\}^{-\lambda}$$

and so to

$$(1-z)[1-(1+x)(1+y)z] = [1-(1+x)z][1-(1+y)z]-xyz.$$

This equation is easily verified.

This completes the proof of (5.1).

The identity (5.1) contains numerous interesting special cases. In particular, taking

$$c_n = \frac{(a)_n}{(c)_n}, \qquad d_n = \frac{(b)_n}{(d)_n},$$

(5.1) becomes

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}z^n\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{(a)_r}{(c)_r}x^r\sum_{s=0}^{n}(-1)^s\binom{n}{s}\frac{(b)_s}{(d)_s}y^s \tag{5.6}$$

$$= (1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{(xyz)}{(1-z)^{2n}}C_n(x,z)D_n(y,z),$$

where now

$$\left\{\begin{aligned} C_n(x,z) &= \sum_{r=0}^{\infty}\frac{(\lambda+n)_r(a)_{n+r}}{r!(c)_{n+r}}\left(\frac{-xz}{1-z}\right)^r\\ D_n(y,z) &= \sum_{s=0}^{\infty}\frac{(\lambda+n)_s(b)_{n+s}}{s!(d)_{n+s}}\left(\frac{-yz}{1-z}\right)^s .\end{aligned}\right. \tag{5.7}$$

This result was proved in an entirely different way by Meixner [5].

We now specialize (5.6) further by taking $\lambda = c = d$. Thus (5.7) reduces to

$$C_n(x,z) = \sum_{r=0}^{\infty}\frac{(a)_{n+r}}{r!}\left(\frac{-xz}{1-z}\right)^r = (a)_n\left(1+\frac{xz}{1-z}\right)^{-a-n} = (a)_n(1-z)^{a+n}(1-(1-x)z)^{-a-n},$$

$$D_n(y,z) = (b)_n(1-z)^{b+n}(1-(1-y)z)^{-b-n}.$$

Therefore (5.6) reduces to

$$\sum_{n=0}^{\infty}\frac{(c)_n}{n!}z^n\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{(a)_r}{(c)_r}x^r\sum_{s=0}^{n}(-1)^s\binom{n}{s}\frac{(b)_s}{(c)_s}y^s$$

$$= (1-z)^{a+b-c}(1-(1-x)z)^{-a}(1-(1-y)z)^{-b}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{xyz}{(1-(1-x)z)(1-(1-y)z)}\right)^n.$$

This is the same as (2.1).

REFERENCES

1. Paul S. Bruckman, "An Interesting Sequence of Numbers Derived from Various Generating Functions," *The Fibonacci Quarterly,* Vol. 10, No. 2 (February 1972), pp. 169–181.
2. Paul S. Bruckman, "Some Generalizations Suggested by Gould's Systematic Treatment of Certain Binomial Identities," *The Fibonacci Quarterly,* Vol. 11, No. 3 (October 1973), pp. 225–240.
3. L. Carlitz, "Some Generating Functions of Weisner," *Duke Math. Journal,* Vol. 28 (1961), pp. 523–529
4. H.W. Gould, "Some Combinatorial Identities of Bruckman. A Systematic Treatment with Relation to the Older Literature," *The Fibonacci Quarterly,* Vol. 10, No. 6 (December 1972), pp. 613–627.
5. J. Meixner, "Umformung gewisser Reihen, deren Glieder Produkte hypergeometrischer Funktionen sind," *Deutsche Math.,* Vol. 6 (1942), pp. 341–489.
6. L. Weisner, "Group-Theoretic Origins of Certain Generating Functions," *Pacific Journal of Math.,* Vol. 5 (1955), pp. 1033–1039.

****