NOTE ON SOME GENERATING FUNCTIONS

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1. In a recent paper in this Quarterly, Bruckman [2] defined a sequence of positive integers B_k by means of

$$(1.1) \qquad (1-x)^{-1}(1+x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} B_k \frac{x^k}{2^k \cdot k!} \quad .$$

This is equivalent to the recurrence

$$(1.2) \qquad B_k = B_{k-1} + (2k-1)(2k-2)B_{k-2} \qquad (k \geqslant 2), \qquad B_0 = B_1 = 1 \ .$$

Making use of (1.2) he showed that

$$(1.3) \qquad e^{x^2/2} \int_0^x e^{-u^2} du = \sum_{k=0}^{\infty} B_k \frac{x^{2k+1}}{(2k+1)!}$$

and

$$(1.4) \qquad (1-x^2)^{-1} \arctan x = \sum_{k=0}^{\infty} B_k^2 \frac{x^{2k+1}}{(2k+1)!} \quad .$$

Bateman [1] has discussed the polynomial $g_n(y,z)$ defined by

$$(1.5) \qquad (1+x)^{y+z}(1-x)^{-y} = \sum_{n=0}^{\infty} x^n g_n(y,z);$$

see also [3]. On the other hand the Jacobi polynomial [6, Ch. 16]

$$(1.6) \qquad P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{a+n}{n-k}\binom{\beta+n}{k}\left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}$$

satisfies

$$(1.7) \qquad \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)z^n = \left(1+\frac{x+1}{2}z\right)^{\alpha}\left(1+\frac{x-1}{2}z\right)^{\beta}$$

and in particular, for $x = 0$,

$$(1.8) \qquad \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(0)z^n = (1+\tfrac{1}{2}z)^{\alpha}(1-\tfrac{1}{2}z)^{\beta} \ .$$

It follows from (1.1) and (1.8) that

$$(1.9) \qquad \frac{1}{k!} B_k = 2^{2k} P_k^{(-\frac{1}{2}-k,-1-k)}(0) = (-1)^k 2^{2k} P_k^{(-1-k,-\frac{1}{2}-k)}(0) \ .$$

We shall show that both (1.3) and (1.4) can be generalized considerably. We also obtain the following congruence for B_n:

(1.10)
$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{n+sm} B_{(r-s)m} \equiv 0 \quad (mod\ r!\ m^r),$$

where m and r are arbitrary positive integers.

It would be of interest to find a combinatorial interpretation of B_k.

2. The writer [4] has obtained the following bilinear generating function:

(2.1)
$$\sum_{n=0}^{\infty} \frac{n!}{(\gamma)_n} (x-1)^n (y-1)^n w^n P_n^{(\alpha-n,-\alpha-\gamma-n)} \left(\frac{x+1}{x-1} \right) P_n^{(\beta-n,-\beta-\gamma-n)} \left(\frac{y+1}{y-1} \right)$$
$$= (1-w)^{-\alpha-\beta-\gamma} (1-xw)^\alpha (1-yw)^\beta F \left[-\alpha, -\beta; \gamma; \frac{(x-1)(y-1)w}{(1-xw)(1-yw)} \right],$$

where as usual

$$F(z,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\ (c)_n} z^n \quad \text{and} \quad (a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1.$$

In particular, for $x = y = -1$ and $\gamma = -\alpha - \beta$, Eq. (2.1) reduces to

(2.2)
$$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} 4^n w^n P_n^{(\alpha-n,\beta-n)}(0) P_n^{(\beta-n,\alpha-n)}(0) = (1+w)^{\alpha+\beta} F \left[-\alpha, -\beta; -\alpha-\beta; \frac{4w}{(1+w)^2} \right].$$

It is convenient to replace α, β by $-\alpha, -\beta$, so that (2.2) becomes

(2.3)
$$\sum_{n=0}^{\infty} \frac{n!}{(\alpha+\beta)_n} 4^n w^n P_n^{(-\alpha-n,-\beta-n)}(0) P_n^{(-\beta-n,-\alpha-n)}(0) = (1+w)^{-\alpha-\beta} F \left[\alpha, \beta; \alpha+\beta; \frac{4w}{(1+w)^2} \right].$$

Specializing further, we take $\beta = \alpha + \frac{1}{2}$, so that

(2.4)
$$\sum_{n=0}^{\infty} \frac{n!}{(2\alpha+\frac{1}{2})_n} 4^n w^n P_n^{(-\alpha-n,-\alpha-\frac{1}{2}-n)}(0) P_n^{(-\alpha-\frac{1}{2}-n,-\alpha-n)}(0)$$
$$= (1+w)^{-2} \alpha^{-\frac{1}{2}} F \left[\alpha, \alpha+\frac{1}{2}; 2\alpha+\frac{1}{2}, \frac{4w}{(1+w)^2} \right].$$

Next in formula (2) of [6, p. 66],

$$F \left[\frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}; a-b+1; \frac{4z}{(1+z)^2} \right] = (1+z)^a F[a,b; a-b+1; z]$$

take $a = 2\alpha$, $b = \frac{1}{2}$. We get

(2.5)
$$F \left[\alpha, \alpha+\frac{1}{2}; 2\alpha+\frac{1}{2}; \frac{4z}{(1+z)^2} \right] = (1+z)^2 F[2\alpha, \frac{1}{2}; 2\alpha+\frac{1}{2}; z].$$

Hence (2.4) becomes

(2.6)
$$\sum_{n=0}^{\infty} \frac{n!}{(2\alpha+\frac{1}{2})_n} 4^n w^n P_n^{(-\alpha-n,-\alpha-\frac{1}{2}-n)}(0) P_n^{(-\alpha-\frac{1}{2}-n,-\alpha-n)}(0) = (1+w)^{-\frac{1}{2}} F[2\alpha, \frac{1}{2}; 2\alpha+\frac{1}{2};w].$$

Since

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x),$$

(2.6) may be written in the form

(2.7)
$$\sum_{n=0}^{\infty} \frac{n!}{(2\alpha+\frac{1}{2})_n} 4^n w^n \left\{ P_n^{(-\alpha-n,-\alpha-\frac{1}{2}-n)}(0) \right\}^2 = (1-w)^{-\frac{1}{2}} F[2\alpha, \frac{1}{2}; 2\alpha+\frac{1}{2}; -w].$$

In particular, for $\alpha = \frac{1}{2}$, it follows from (2.7) and (1.9) that

$$\sum_{n=0}^{\infty} \frac{2^{-2n} w^n}{n!(3/2)_n} B_n^2 = (1-w)^{-\frac{1}{2}} F[1,\frac{1}{2}; 3/2; -w].$$

Replacing w by z^2, this becomes

(2.8)
$$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} B_n^2 = z(1-z^2)^{-\frac{1}{2}} F[1, \frac{1}{2}; 3/2; -z^2] .$$

Since

$$zF[1, \frac{1}{2}; 3/2; -z^2] = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)_n}{(3/2)_n} z^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} = \arctan z ,$$

it is evident that (2.8) is the same as (1.4).

3. In (2.1) take $x = -1$, $y = 0$, $\gamma = -a - \beta$. Since, by (1.6),
$$P_n^{(\beta-n,\alpha-n)}(-1) = \binom{a}{n} ,$$
it is clear that (2.1) reduces to

$$\sum_{n=0}^{\infty} \frac{n!}{(-a-\beta)_n} \binom{a}{n} 2^n w^n P_n^{(\alpha-n,\beta-n)}(0) = (1+w)^a F\left[-a, -\beta; -a-\beta; \frac{2w}{1+w} \right] .$$

Replacing a, β by $-a, -\beta$, this becomes

(3.1)
$$\sum_{n=0}^{\infty} (-1)^n \frac{(a)_n}{(a+\beta)_n} 2^n w^n P_n^{(-\alpha-n,-\beta-n)}(0) = (1+w)^{-a} F\left[a, \beta; a+\beta; \frac{2w}{1+w} \right] .$$

In particular, for $\beta = \frac{1}{2}$, we get

(3.2)
$$\sum_{n=0}^{\infty} \frac{(a)_n}{(a+\frac{1}{2})_n} 2^n z^n P_n^{(-\alpha-n,-\frac{1}{2}-n)}(0) = (1-z)^{-a} F\left[a, \frac{1}{2}; a+\frac{1}{2}; \frac{-2z}{1-z} \right] .$$

For $a = 1$, Eq. (3.2) becomes

(3.3)
$$\sum_{n=0}^{\infty} \frac{z^n}{2^n(3/2)_n} B_k = (1-z)^{-1} F\left[1, \frac{1}{2}; 3/2; - \frac{2z}{1-z} \right] .$$

This is not the same as (1.3).

The right-hand side of (3.2) is equal to

$$\sum_{r=0}^{\infty} \frac{(a)_r(\frac{1}{2})_r}{r!(a+\frac{1}{2})_r} (-2z)^r (1-z)^{-\alpha-r} = \sum_{r=0}^{\infty} \frac{(a)_r(\frac{1}{2})_r}{r!(a+\frac{1}{2})_r} (-2z)^r \sum_{s=0}^{\infty} \frac{(a+r)_s}{s!} z^s$$

$$= \sum_{n=0}^{\infty} (a)_n z^n \sum_{r=0}^{n} (-2)^r \frac{(\frac{1}{2})_r}{r!(n-r)!(a+\frac{1}{2})_r} .$$

Hence (3.2) implies

$$\sum_{n=0}^{\infty} \frac{2^n z^n}{(+\frac{1}{2})_n} P_n^{(-\alpha-n,-\frac{1}{2}-n)}(0) = \sum_{n=0}^{\infty} z^n \sum_{r=0}^{n} (-2)^r \frac{(\frac{1}{2})_r}{r!(n-r)!(a+\frac{1}{2})_r} = \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r(-2z)^r}{r!(a+\frac{1}{2})_r} \sum_{n=r}^{\infty} \frac{z^{n-r}}{(n-r)!} ,$$

so that

(3.4)
$$\sum_{n=0}^{\infty} \frac{z^n}{(a+\frac{1}{2})_n} P_n^{(-\alpha-n,-\frac{1}{2}-n)}(0) = e^{\frac{1}{2}z} \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r(-z)^r}{r!(a+\frac{1}{2})_r} .$$

For $a = 1$, Eq. (3.4) reduces to (1.3) .

4. Put

(4.1)
$$(1-x)^{\alpha}(1+x)^{\beta} = \sum_{n=0}^{\infty} c_n(a,\beta)x^n .$$

Then

$$\sum_{m,n=0}^{\infty} c_m(\alpha,\beta)c_n(\alpha,\beta)x^m y^n = (1-x)^{\alpha}(1-y)^{\alpha}(1+x)^{\beta}(1+y)^{\beta} = (1+xy-x-y)^{\alpha}(1+xy+x+y)^{\beta}$$

$$= (1+xy)^{\alpha+\beta} \left(1 - \frac{x+y}{1+xy} \right)^{\alpha} \left(1 + \frac{x+y}{1+xy} \right)^{\beta}$$

$$= (1+xy)^{\alpha+\beta} \sum_{k=0}^{\infty} c_k(\alpha,\beta) \left(\frac{x+y}{1+xy} \right)^k$$

$$= \sum_{k=0}^{\infty} c_k(\alpha,\beta) \sum_{s=0}^{k} \binom{k}{s} x^s y^{k-s} \sum_{r=0}^{\infty} \binom{\alpha+\beta-k}{t} x^t y^t$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_{\substack{s+t=m \\ k-s+t=n}} \binom{k}{s}\binom{\alpha+\beta-k}{t} c_k(\alpha,\beta).$$

It follows that

(4.2) $$c_m(\alpha,\beta)c_n(\alpha,\beta) = \sum_{t=0}^{min(m,n)} \binom{m+n-2t}{m-t}\binom{\alpha+\beta-m-n+2t}{t} c_{m+n-2t}(\alpha,\beta).$$

The proof follows Kaluza [6]; see also [3].

Comparing (4.1) with (1.1), we have

(4.3) $$B_k = 2^k \cdot k! c_k(-1,-\tfrac{1}{2}).$$

Thus (4.2) implies

(4.4) $$B_m B_n = \sum_{t=0}^{min(m,n)} (-1)^t 2^t \binom{m}{t}\binom{n}{t} t! \prod_{j=0}^{t-1} (2m+2n-2t-2j+1)B_{m+n-2t}.$$

For $m = 1$, Eq. (4.4) reduces to (1.2). It is not difficult to prove (4.4) by induction.

The writer has proved the following result [5].

Let $f(n)$, $g(n)$ denote polynomials in n with integral coefficients. Define u_n by means of

(4.5) $$u_{n+1} = f(n)u_n + g(n)u_{n-1}   (n \geqslant 1),$$
where
(4.6) $$u_0 = 1,   u_1 = f(0),   g(0) = 0.$$

Then u_n satisfies the following congruence:

(4.7) $$\Delta^{2r}u_n \equiv \Delta^{2r-1}u_n \equiv 0   (mod\ m^r),$$

for all $m \geqslant 1$, $n \geqslant 0$, $r \geqslant 1$, where

$$\Delta^r u_n = \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} u_{n+sm}u_{(r-s)m}.$$

Comparing (4.5) with

$$B_{n+1} = B_n + 2n(2n+1)B_{n-1},$$

it is clear that (4.6) holds. We have therefore

(4.8) $$\sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} B_{n+sm}B_{(r-s)m} \equiv 0   (mod\ m^{[(r+1)/2]}).$$

However a better result can be obtained. By (4.4) we have

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{n+sm} B_{(r-s)m} = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \sum_{t} (-1)^{t} 2^{t} \binom{n+sm}{t} \binom{(r-s)m}{t} t! \cdot \prod_{j=0}^{t-1}$$

$$\cdot (2n + 2rm - 2t - 2j + 1) \cdot B_{n+rm-2t} = \sum_{t} (-1)^{t} \frac{2^{t}}{t!} B_{n+rm-2t} \prod_{j=0}^{t-1} (2n + 2rm - 2t - 2j + 1)$$

$$\cdot \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f(s),$$

where

$$f(s) = (n + sm - t + 1)_t ((r - s)m - t + 1)_t .$$

Clearly

$$f(s) = a_0 + a_1 sm + \cdots + a_{2t}(sm)^{2t} ,$$

where the a_i are integers. Then

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f(s) = \sum_{i=r}^{2t} a_i m^{2i} \Delta^r 0^i \equiv 0 \pmod{r! m^r} .$$

Since

$$\frac{2^t}{t!} \prod_{j=0}^{t-1} (2n + 2rm - 2t - 2j + 1)$$

is integral, it follows at once that

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{n+sm} B_{(r-s)m} \equiv 0 \pmod{r! m^r} .$$

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