GENERALIZED FIBONACCI TILING

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1. INTRODUCTION

One way to easily establish the validity of

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}$$

is by use of a nice geometric argument as in Brother Alfred [1]. Thus by starting with two unit squares one can add a whirling array of squares (see Fig. 1) with Fibonacci number sides since the area as in Fig. 1 is

$$5 \times 8 = F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 = F_5F_6.$$

More generally, the rectangle has area F_nF_{n+1}.

This result is classic, but a new twist was added by H. L. Holden [2]. The centers of the outwardly spirally squares lie on two straight lines which are orthogonal. These two straight lines intersect in a point P, and the distances of the centers of the squares from P sequentially are proportional to the Lucas numbers. Holden also contains an extension to the generalized Fibonacci sequence with $H_1 = 1$ and $H_2 = p$ with $H_{n+2} = H_{n+1} + H_n$. This results in

$$H_1^2 + H_2^2 + H_3^2 + \cdots + H_n^2 = H_nH_{n+1} - H_0 .$$

In another paper, we will discuss the situation with inwinding spirals.

2. THE FIRST GENERALIZATION

Our method here is different than that used by Holden [2], but ours offers a neater way to get the centers of the squares, and we proceed principally by generating functions. (See Fig. 2.) We first discuss the geometry.

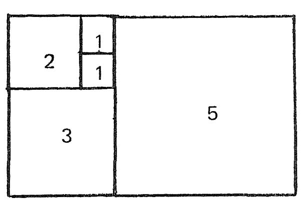

Figure 1

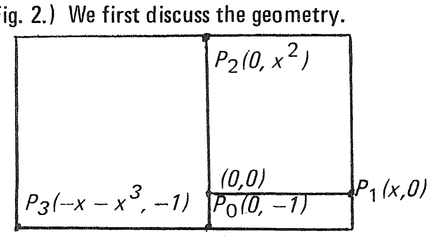

Figure 2

Start out with a unit square and make x copies. Then above that, make x copies of $x \cdot x$ squares. It is not difficult to see that the edges are $1, x, x^2 + 1, x^3 + 2x, \cdots, f_n(x)$, where $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, which are the Fibonacci polynomials.

If we consider the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

as in Holden and $V_1 = (x, 1)$, $V_2 = (-1, x)$, $V_3 = (-x, -1)$, $V_4 = (1, -x)$ with

$$V_{n+1} = V_n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

then

Theorem 1. $$P_n = P_{n-1} + V_nf_n(x).$$

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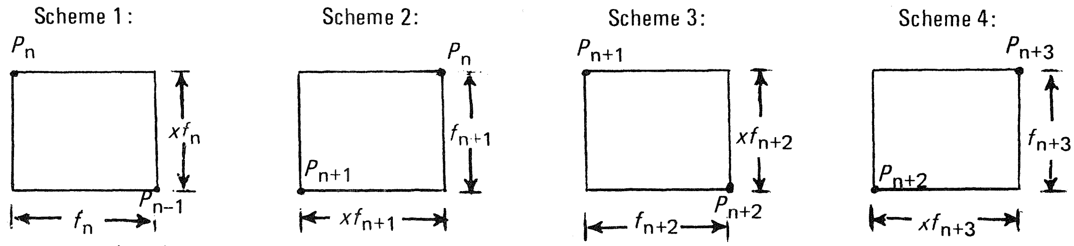
Proof. The proof proceeds in four parts.



$P_n = P_{n-1} + (-1,x)f_n$ $\quad$ $P_{n+1} = P_n + (-x,-1)f_{n+1}$ $\quad$ $P_{n+2} = P_{n+1} + (1,-x)f_{n+2}$ $\quad$ $P_{n+3} = P_{n+2} + (x,1)f_{n+3}$

These are the four critical turns in the sequence of expanding the outward spiralling of squares.

As a consequence of Theorem 1, one can prove

Theorem 2.

$$P_n = P_0 + \sum_{i=1}^{n} V_i f_i(x) .$$

Proof.

$$P_1 = (0,-1) + (x,1)\cdot 1 = (x,0).$$

Assume

$$P_n = P_{n-1} + V_n f_n = P_0 + \sum_{i=1}^{n-1} V_i f_i + V_n f_n = P_0 + \sum_{i=1}^{n} V_i f_i .$$

We are now ready to get on with the general theorem by means of generating functions.

3. SOME NECESSARY IDENTITIES

Lemma 1.

$$\frac{\lambda x}{1-\lambda^2(x^2+2)+\lambda^4} = \sum_{n=0}^{\infty} f_{2n}(x)\lambda^{2n+1}$$

Lemma 2.

$$\frac{(x^2+1)-\lambda^2}{1-\lambda^2(x^2+2)+\lambda^4} = \sum_{n=0}^{\infty} f_{2n+3}(x)\lambda^{2n}$$

Lemma 3.

$$\frac{\lambda+\lambda^3}{1-\lambda^2(x^2+2)+\lambda^4} = \sum_{n=0}^{\infty} f_{2n+1}(x)\lambda^{2n+1}$$

Since these are straightforward, the proofs will be omitted.

We may now give a generating function for the x-components of the corners, where $P_{n,x}$ denotes the x-coordinate of the point P_n.

Theorem 3.

$$\sum_{i=0}^{\infty} P_{i,x}\lambda^i = \frac{x(\lambda-\lambda^2+\lambda^3)}{1+\lambda^2(x^2+2)+\lambda^4}\cdot\frac{1}{1-\lambda}$$

Proof. From

$$P_n = (0,-1) + \sum_{i=1}^{n} V_i f_i(x),$$

$$\sum_{n=0}^{\infty} P_{n,x}\lambda^n = (0 + xf_1\lambda - f_2\lambda^2 - xf_3\lambda^3 + f_4\lambda^4 + xf_5\lambda^5 + \cdots)/(1-\lambda)$$
$$= [x(f_1\lambda - f_3\lambda^3 + f_5\lambda^5 - \cdots) - (f_2\lambda^2 - f_4\lambda^4 + f_6\lambda^6 - \cdots)]/(1-\lambda)$$
$$= \frac{\lambda x + x(\lambda + \lambda^3)}{1 + \lambda^2(x^2+2) + \lambda^4} \cdot \frac{1}{1-\lambda} .$$

Since P_n and P_{n-1} are opposite corners of square n, the x-coordinates of the centers C_n are given by

$$(P_{n,x} + P_{n-1,x})/2 = C_{n,x} .$$

$$\sum_{n=1}^{\infty} \lambda^n (P_{n,x} + P_{n-1,x})/2 = \frac{1+\lambda}{2(1-\lambda)} \cdot \frac{x(\lambda - \lambda^2 + \lambda^3)}{1 + \lambda^2(x^2+2) + \lambda^4} = \sum_{i=1}^{\infty} C_{i,x}\lambda^i$$

and

$$\sum_{n=1}^{\infty} (C_{n+2,x} - C_{n,x})\lambda^{n+1} = \frac{(1+\lambda)^2[x(\lambda - \lambda^2 + \lambda^3)]}{2(1 + \lambda^2(x^2+2) + \lambda^4)} + \frac{\lambda}{2} - \frac{\lambda^2 x}{2}$$
$$= \frac{-\lambda^3 x(x^2+2)/2 - \lambda^4 x(x^2+1+\lambda^2)/2}{1 + \lambda^2(x^2+2) + \lambda^4} ,$$

where

$$C_{1,x} = x/2 \qquad \text{and} \qquad C_{2,x} = x/2 .$$

There are two further things to do. These differences clearly alternate in sign. To convert to regular differences all the same sign, we change the minus sign to a plus in front of the even powered term and then replace λ^2 by $-\lambda^2$. This results in the following theorem:

Theorem 4.

$$G_x(x,\lambda) = \frac{-\lambda^3 x(x^2+2)/2 + \lambda^4 x(x^2+1-\lambda^2)/2}{1 - \lambda^2(x^2+2) + \lambda^4} .$$

Theorem 5. The generating function for the y-differences between alternate corners is

$$\frac{\lambda^3 x(x/2) + \lambda^4(x^2+1-\lambda^2)[(x^2+2)/2]}{1 - \lambda^2(x^2+2) + \lambda^4} .$$

Proof. For the y-differences one begins with

$$\sum_{i=0}^{\infty} P_{i,y}\lambda^i = \frac{1}{1-\lambda}\left[-1 + \frac{\lambda + \lambda^2 x^2 + \lambda^3}{1 + \lambda^2(x^2+2) + \lambda^4}\right]$$

and

$$\sum_{i=1}^{\infty} P_{i,y}\lambda^i = \frac{1}{1-\lambda}\left[-\lambda + \frac{\lambda + \lambda^2 x^2 + \lambda^3}{1 + \lambda^2(x^2+2) + \lambda^4}\right] .$$

$$\sum_{i=1}^{\infty} C_{i,y}\lambda^i = \sum_{i=1} \lambda^i (P_{i,y} + P_{i-1,y})/2 = \tfrac{1}{2}\left\{\frac{\lambda}{1-\lambda}\left[-1 + \frac{\lambda + \lambda^2 x^2 + \lambda^3}{1 + \lambda^2(x^2+2) + \lambda^4}\right] + \frac{1}{1-\lambda}\left[-\lambda + \frac{\lambda + \lambda^2 x^2 + \lambda^3}{1 + \lambda^2(x^2+2) + \lambda^4}\right]\right\}$$
$$= \frac{1}{2(1-\lambda)} \cdot \frac{-\lambda + \lambda^2(x^2+1) - \lambda^3(x^2+3) + \lambda^4 - 2\lambda^5}{1 + \lambda^2(x^2+2) + \lambda^4} .$$

Now to directly form the y-differences:

$$\sum_{i=1}^{\infty} (C_{i+2,y} - C_{i,y})\lambda^{i+2} = (1-\lambda^2)\sum_{i=1}^{\infty} C_{i,y}\lambda^i + C_{1,y}\lambda + C_{2,y}\lambda^2 .$$

But, $C_{1,y} = -½$ and $C_{2,y} = x^2/2$. Thus,

$$\sum_{i=1}^{\infty} (C_{i+2,y} - C_{i,y})\lambda^i = \frac{1+\lambda}{2} \cdot \frac{-\lambda + \lambda^2(x^2+1) - \lambda^3(x^2+3) + \lambda^4 - 2\lambda^5}{1+\lambda^2(x^2+2)+\lambda^4}$$

$$= \frac{\lambda^3x^2 - \lambda^4(x^4+3x^2+2) - (x^2+2)\lambda^6}{2(1+\lambda^2(x^2+2)+\lambda^4)} .$$

From the diagrams it is clear that these differences are associated with odds and evens and on their respective lines they alternate in sign. We wish the initial values to be positive.

$$\sum_{i=1}^{\infty} |C_{i+2,y} - C_{i,y}|\lambda^i = -\left[\frac{(-\lambda^2)\lambda x^2 + (-\lambda)^2(x^4+3x^2+2) + (x^2+2)(-\lambda^2)^3}{1-\lambda^2(x^2+2)+\lambda^4}\right]$$

$$= \frac{\lambda^3 x(x/2) + \lambda^4(x^2+1-\lambda^2)[(x^2+2)/2]}{1-\lambda^2(x^2+2)+\lambda^4}$$

$$= \frac{x}{2}\sum_{n=0}^{\infty} f_{2n}(x)\lambda^{2n+1} + \frac{x^2+2}{2}\sum_{n=0}^{\infty} f_{2n+3}(x)\lambda^{2n} .$$

Recall that the x-differences

$$\sum_{i=1}^{\infty} (-1)^i|C_{i+2,x} - C_{i,x}|\lambda^i = -\frac{x^2+2}{2}\sum_{n=0}^{\infty} f_{2n}(x)\lambda^{2n+1} + \frac{x}{2}\sum_{n=0}^{\infty} f_{2n+3}(x)\lambda^{2n} .$$

Thus, uniformly we see that the slopes of the lines through the centers are

$$\frac{C_{n+2,y} - C_{n,y}}{C_{n+2,x} - C_{n,x}} = \frac{x/2}{-(x^2+2)/2} = \frac{-x}{x^2+2}, \quad \text{odd } n;$$

$$\frac{C_{n+2,y} - C_{n,y}}{C_{n+2,x} - C_{n,x}} = \frac{(x^2+2)/2}{(x/2)} = \frac{x^2+2}{x}, \quad \text{even } n .$$

Thus, the centers lie on two straight lines which are orthogonal since the product of their slopes is −1. This concludes the proof.

Theorem 6. The centers C_{2i+1} lie on a line with slope $-x/(x^2+2)$ and the centers C_{2i} lie on a line with slope $(x^2+2)/x$. These lines are orthogonal and intersect at the point (u,v), where

$$u = \frac{x}{x^2+4} \quad \text{and} \quad v = \frac{-2}{x^2+4} .$$

Proof. It is easy to show that the lines through C_1 and C_3, and through C_2 and C_4, respectively, do meet in the point (u,v) specified.

Theorem 7. If Q is the point

$$\left(\frac{x}{x^2+4}, \frac{-2}{x^2+4}\right),$$

then the center C_n is D_n units from Q, where

$$D_n = \frac{\mathcal{L}_n(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)}$$

and $\mathcal{L}_n(x)$ is the n^{th} Lucas polynomial, $\mathcal{L}_1(x) = x$, $\mathcal{L}_2(x) = x^2+2$, and $\mathcal{L}_{n+1}(x) = x\mathcal{L}_n(x) + \mathcal{L}_{n-1}(x)$.

Proof. Given that $C_1 = (x/2, -1/2)$ and $C_2 = (x/2, x^2/2)$, one can compute

$$D_1^2 = \left(\frac{x}{x^2+4} - \frac{x}{2}\right)^2 + \left(\frac{-2}{x^2+4} + \frac{1}{2}\right)^2 = \frac{x^2(x^2+2)^2+x^4}{[2(x^2+4)]^2} = \frac{x^6+5x^4+4x^2}{[2(x^2+4)]^2}$$

$$D_1 = \frac{x\sqrt{x^4+5x^2+4}}{2(x^2+4)} .$$

It is also easy to verify that

$$D_2 = \frac{(x^2+2)\sqrt{x^4+5x^2+4}}{2(x^2+4)} .$$

Now consider the centers C_{n+2} and C_n. The points lie on a line through

$$\left(\frac{x}{x^2+4}, \frac{-2}{x^2+4}\right)$$

which separates them. The x and y differences from C_{n+2} and C_n are

$$-\frac{x^2+2}{2} f_n(x) \quad \text{and} \quad \frac{x}{2} f_n(x),$$

respectively, for one line or

$$\frac{x}{2} f_n(x) \quad \text{and} \quad \frac{x^2+2}{2} f_n(x),$$

respectively, for the second line. Thus the distance

$$|C_{n+2} - C_n| = \sqrt{x^4+5x^2+4}\, f_{n+1}(x)/2$$

in any case.

There is an identity for Lucas polynomials (see [3], p. 82)

$$\mathcal{L}_{n+1}(x) + \mathcal{L}_{n-1}(x) = (x^2+4)f_n(x).$$

Now, suppose

$$D_n = \frac{\mathcal{L}_n(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)} .$$

Then

$$|C_{n+2} - C_n| - D_n = D_{n+2}$$

$$\frac{f_{n+1}(x)\sqrt{x^4+5x^2+4}}{2} - \frac{\mathcal{L}_n(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)} = \frac{\mathcal{L}_{n+2}(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)} .$$

This concludes the proof of Theorem 7.

3. THE SECOND GENERALIZATION

In the last section we considered the rectangle whose edges were $f_n(x)$ and $xf_n(x)$, where

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$$

with

$$f_0(x) = 0 \quad \text{and} \quad f_1(x) = 1;$$

that is, the Fibonacci polynomials. Here we consider the sequence of polynomials such that

$$U_1(x) = 1, \qquad U_2(x) = P, \qquad \text{and} \qquad U_{n+2}(x) = xU_{n+1}(x) + U_n(x),$$

the generalized Fibonacci polynomials. We shall prove the following theorem.

Theorem 8. If one starts with a $1 \times p$ rectangle and adds counter-clockwise rectangles p by $px, \cdots, U_n(x)$ by $xU_n(x)$, then such squares in the whirling array have their centers on two straight lines with slopes $-(x^2+2)/x$ and $x/(x^2+2)$, which are orthogonal.

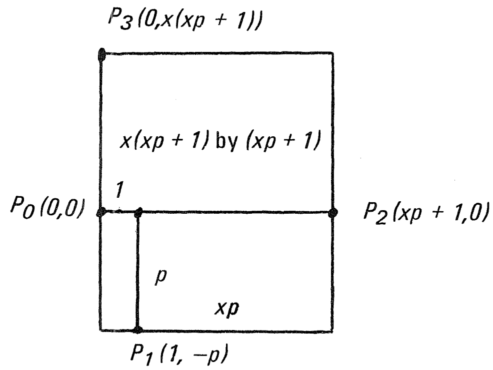


Figure 3

Proof. To establish that the centers lie on two perpendicular straight lines we shall have to find the coordinates of the vertices P_n. As before, we consider the rotation matrix

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the sequence of vectors

$$V_1^* = (1,-x), \qquad V_2^* = (x,1), \qquad V_3^* = (-1,x), \qquad V_4^* = (-x,-1),$$

where $V_n^* = V_m^*$ when $n \equiv m \pmod 4$.

We shall also need the following identities for Fibonacci and Lucas polynomials (see [3]):

$$f_{n+2k}(x) - f_{n-2k}(x) = \mathcal{L}_n(x)f_{2k}(x)$$

$$f_{n+2k}(x) + f_{n-2k}(x) = f_n(x)\mathcal{L}_{2k}(x)$$

$$f_{n+2k+1}(x) + f_{n-2k-1}(x) = \mathcal{L}_n(x)f_{2k+1}(x)$$

$$f_{n+2k+1}(x) - f_{n-2k-1}(x) = f_n(x)\mathcal{L}_{2k+1}(x)\,.$$

It is then easy to establish, on lines similar to Theorem 1, that

(1) $$P_n = P_{n-1} + V_n U_n(x),$$

where the $\{U_n\}$ is the sequence of polynomials

(2) $$U_1(x) = 1, \qquad U_2(x) = p, \qquad U_n(x) = xU_{n-1}(x) + U_{n-2}(x).$$

One also recognizes from (2) that the U_n obey

(3) $$U_{n+1}(x) = pf_n(x) + f_{n-1}(x)\,,$$

where the $f_n(x)$ are the standard Fibonacci polynomials.

If $P_{n,x}$ and $P_{n,y}$ denote the x- and y-coordinates of P_n, one can establish from (1) that

$$\begin{aligned} P_{2n+1,x} &= (U_1 - U_3 + U_5 - \cdots + (-1)^n U_{2n+1}) + (-x)(-U_2 + U_4 - \cdots + (-1)^n U_{2n}) \\ &= (-1)^n f_{n+1}(x)U_{n+1}(x) + (-1)^{n+1} x f_n(x)U_{n+1}(x) = (-1)^n U_{n+1}(x)f_{n-1}(x) \end{aligned}$$

from which one can easily deduce that

$$P_{2n+3,x} = (-1)^{n+1}U_{n+2}(x)f_n(x), \qquad P_{2n-1,x} = (-1)^{n-1}U_n(x)f_{n-2}(x).$$

We also have

$$\begin{aligned} P_{2n,x} &= [U_1(x) - U_3(x) + U_5(x) - \cdots + (-1)^{n-1}U_{2n-1}(x)] \\ &\quad -x[-U_2(x) + U_4(x) - \cdots + (-1)^n U_{2n}(x)] \\ &= (-1)^{n-1}f_n(x)U_n(x) + (-1)^{n+1}xf_n(x)U_{n+1}(x) \\ &= (-1)^{n-1}f_n(x)U_{n+2}(x) \end{aligned}$$

which implies that

$$\begin{aligned} P_{2n+2,x} &= (-1)^n f_{n+1}(x)U_{n+3}(x), \\ P_{2n-2,x} &= (-1)^n f_{n-1}(x)U_{n+1}(x). \end{aligned}$$

Now, because the C_n are the centers of the squares, we get

$$C_n = (P_n + P_{n-1})/2$$

and so we get

$$\begin{aligned} C_{2n+3,x} - C_{2n+1,x} &= (P_{2n+3,x} + P_{2n+2,x} - P_{2n+1,x} - P_{2n,x})/2 \\ &= \tfrac{1}{2}[(-1)^{n+1}U_{n+2}(x)f_n(x) + (-1)^n f_{n+1}(x)U_{n+3}(x) \\ &\quad + (-1)^{n+1}U_{n+1}(x)f_{n-1}(x) + (-1)^n f_n(x)U_{n+2}(x)] \\ &= \frac{(-1)^n}{2}\,[(pf_{n+2}(x) + f_{n+1}(x))f_{n+1}(x) - (pf_n(x) + f_{n-1}(x))f_{n-1}(x)] \\ &= \frac{(-1)^n}{2}\,[p(f_{n+2}(x)f_{n+1}(x) - f_n(x)f_{n-1}(x)) + f^2_{n+1}(x) - f^2_{n-1}(x)] \\ &= \frac{(-1)^n}{2}\,[pxf_{2n+1}(x) + xf_{2n}(x)] = \frac{(-1)^n}{2}\,xU_{2n+2}(x). \end{aligned}$$

We have also

$$\begin{aligned} C_{2n+2,x} - C_{2n,x} &= (P_{2n+2,x} + P_{2n+1,x} - P_{2n,x} - P_{2n-1,x})/2 \\ &= \tfrac{1}{2}[(-1)^n f_{n+1}(x)U_{n+3}(x) + (-1)^n U_{n+1}(x)f_{n-1}(x) \\ &\quad + (-1)^n f_n(x)U_{n+2}(x) + (-1)^n U_n(x)f_{n-2}(x)] \\ &= \frac{(-1)^n}{2}\,[(pf_{n+2}(x) + f_{n+1}(x))f_{n+1}(x) + (pf_n(x) + f_{n-1}(x))f_{n-1}(x) \\ &\quad + (pf_{n+1}(x) + f_n(x))f_n(x) + (pf_{n-1}(x) + f_{n-2}(x))f_{n-2}(x)] \\ &= \frac{(-1)^n}{2}\,[pf_{n+1}(x)(f_{n+2}(x) + f_n(x)) + pf_{n-1}(x)(f_n(x) + f_{n-2}(x)) \\ &\quad + f^2_{n-2}(x) + f^2_{n-1}(x) + f^2_n(x) + f^2_{n+1}(x)] \\ &= \frac{(-1)^n}{2}\,[pf_{2n+2}(x) + pf_{2n-2}(x) + f_{2n+1}(x) + f_{2n-3}(x)] \\ &= \frac{(-1)^n}{2}\,[p\mathfrak{L}_2(x)f_{2n}(x) + \mathfrak{L}_2(x)f_{2n-1}(x)] = \frac{(-1)^n}{2}\,\mathfrak{L}_2(x)U_{2n+1}(x). \end{aligned}$$

Now we shift our attention to the y-coordinates. From (1) we get

$$\begin{aligned} P_{2n+1,y} &= (-x)(U_1 - U_3 + \cdots + (-1)^n U_{2n+1}) \\ &\quad - (-U_2 + U_4 - U_6 + \cdots + (-1)^n U_{2n}) \\ &= (-1)^{n+1}xf_{n+1}(x)U_{n+1}(x) + (-1)^{n+1}f_n(x)U_{n+1}(x) \\ &= (-1)^{n+1}U_{n+1}(x)f_{n+2}(x). \end{aligned}$$

and

$$\begin{aligned} P_{2n+3,y} &= (-1)^n U_{n+2}(x)f_{n+3}(x), \\ P_{2n-1,y} &= (-1)^n U_n(x)f_{n+1}(x). \end{aligned}$$

We also have

$$\begin{aligned} P_{2n,y} &= (-x)[U_1(x) - U_3(x) + \cdots + (-1)^{n-1}U_{2n-1}(x)] \\ &\quad - [-U_2(x) + U_4(x) - \cdots + (-1)^n U_{2n}(x)] \\ &= (-1)^n x f_n(x)U_n(x) + (-1)^{n+1}f_n(x)U_{n+1}(x) \\ &= (-1)^{n+1}f_n(x)U_{n-1}(x) \end{aligned}$$

and

$$\begin{aligned} P_{2n+2,y} &= (-1)^n f_{n+1}(x)U_n(x), \\ P_{2n-2,y} &= (-1)^n f_{n-1}(x)U_{n-2}(x). \end{aligned}$$

From the above calculations, we find

$$C_{2n+3,y} - C_{2n+1,y} = (P_{2n+3,y} + P_{2n+2,y} - P_{2n+1,y} - P_{2n,y})/2$$

$$\begin{aligned} C_{2n+3,y} - C_{2n+1,y} &= \tfrac{1}{2}[(-1)^n U_{n+2}(x)f_{n+3}(x) + (-1)^n f_{n+1}(x)U_n(x) \\ &\quad + (-1)^n U_{n+1}(x)f_{n+2}(x) + (-1)^n f_n(x)U_{n-1}(x)] \\ &= \frac{(-1)^n}{2}[(pf_{n+1}(x) + f_n(x))f_{n+3}(x) \\ &\quad + (pf_{n-1}(x) + f_{n-2}(x))f_{n+1}(x) + (pf_n(x) + f_{n-1}(x))f_{n+2}(x) \\ &\quad + (pf_{n-2}(x) + f_{n-3}(x))f_n(x)] \\ &= \frac{(-1)^n}{2}[pf_{n+1}(x)(f_{n+3}(x) + f_{n-1}(x)) \\ &\quad + pf_n(x)(f_{n+2}(x) + f_{n-2}(x)) + f_n(x)(f_{n+3}(x) + f_{n-3}(x)) \\ &\quad + f_{n-2}(x)f_{n+1}(x) + f_{n-1}(x)f_{n+2}(x)] \\ &= \frac{(-1)^n}{2}[pf_{n+1}^2(x)\mathcal{L}_2(x) + pf_n^2(x)\mathcal{L}_2(x) \\ &\quad + f_n(x)\mathcal{L}_n(x)f_3(x) + f_{2n}(x)] \\ &= \frac{(-1)^n}{2}\mathcal{L}_2(x)U_{2n+2}(x). \end{aligned}$$

Our final step is to find

$$\begin{aligned} C_{2n+2,y} - C_{2n,y} &= (P_{2n+2,y} + P_{2n+1,y} - P_{2n,y} - P_{2n-1,y})/2 = \tfrac{1}{2}[(-1)^n f_{n+1}(x)U_n(x) \\ &\quad + (-1)^{n+1}U_{n+1}(x)f_{n+2}(x) + (-1)^n f_n(x)U_{n-1}(x) + (-1)^{n+1}U_n(x)f_{n+1}(x)] \\ &= \frac{(-1)^{n+1}}{2}[(pf_n(x) + f_{n-1}(x))f_{n+2}(x) - (pf_{n-2}(x) + f_{n-2}(x))f_n(x)] \\ &= \frac{(-1)^{n+1}}{2}[pf_n(x)(f_{n+2}(x) - f_{n-2}(x)) + f_{n+2}(x)f_{n-1}(x) - f_n(x)f_{n-3}(x)] \\ &= \frac{(-1)^{n+1}}{2}[pf_n(x)\mathcal{L}_n(x)f_2(x) + xf_{2n-1}(x)] = \frac{(-1)^{n+1}}{2}xU_{2n+1}(x). \end{aligned}$$

So, from the above results, we have

$$\frac{C_{2n+3,y} - C_{2n+1,y}}{C_{2n+3,x} - C_{2n+1,x}} = \frac{(-1)^n \mathcal{L}_2(x)U_{2n+2}(x)\cdot 2}{2\cdot U_{2n+1}(x)\cdot x(-1)^n} = \frac{\mathcal{L}_2(x)}{x}$$

which tells us that the C_n for odd n lie on a line with slope $(x^2+2)/2$. We also find

$$\frac{C_{2n+2,y} - C_{2n,y}}{C_{2n+2,x} - C_{2n,x}} = \frac{(-1)^{n+1}xU_{2n+1}(x)\cdot 2}{2\cdot(-1)^n \mathcal{L}_2(x)U_{2n+1}(x)} = -\frac{x}{\mathcal{L}_2(x)}$$

which tells us that the C_n for even n lie on a line with slope $-x/(x^2+2)$. Further, since the products of the slopes is -1, these lines are perpendicular. This proves Theorem 8.

From the above result it follows almost trivially that

Theorem 9. If D_n is the distance of C_n from the point of intersection of the two lines of centers, then

$$D_n = \frac{\mathcal{L}_n^*(x)\sqrt{x^4+5x^2+4}}{2(x^2+4)} ,$$

where the $\mathcal{L}_n^*(x)$ are the generalized Lucas polynomials

$$\mathcal{L}_1^* = p, \qquad \mathcal{L}_2^* = xp+2, \qquad \text{and} \qquad \mathcal{L}_{n+2}^*(x) = x\mathcal{L}_{n+1}^*(x) + \mathcal{L}_n^*(x).$$

REFERENCES

1. Brother Alfred Brousseau, "Fibonacci Numbers and Geometry," *The Fibonacci Quarterly,* Vol. 10, No. 3 (April 1972), pp. 303–318.
2. Herbert L. Holden, "Fibonacci Tiles," *The Fibonacci Quarterly,* Vol. 13, No. 1 (February 1975), pp. 45–49.
3. Ellen King, "Some Fibonacci Inverse Trigonometry," Master's Thesis, San Jose State University, July, 1969, pp. 82–90.

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A LEAST INTEGER SEQUENCE INVESTIGATION

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In the fall semester of 1964, four students, Robert Lera, Ron Staszkow, Rod Arriaga, and Robert Martel began an investigation along with their teacher, Brother Alfred Brousseau, of a problem that arose in connection with a Putnam examination question. The problem was to prove that if

$$P_{n+1} = [p_n + p_{n-1} + p_{n-2}]/p_{n-3}$$

produced an endless sequence of integers while the quantities p_i remained less in absolute value than an upper bound A, then the sequence must be periodic. The divergent idea that led to the research was this: How can one insure an infinite sequence of integers from such a recursion formula? One quick answer was to use the greatest integer function.

Initially an investigation was begun on:

$$a_{n+1} = \left[\frac{a_n + a_{n-1}}{a_{n-2}}\right] ,$$

where the square brackets mean: "take the greatest integer less than or equal to the quantity enclosed within the brackets." Very quickly, zero entered into the sequence with the result that there were mathematical complications once it arrived at the denominator.

To avoid this problem, it was decided to try using "the least integer function" instead of the greatest integer function. The notation adopted was:

$$[x]^* = n ,$$

where n is the least integer greater than or equal to x. With this approach starting with three positive integers the function:

$$a_{n+1} = \left[\frac{a_n + a_{n-1}}{a_{n-2}}\right]^*$$

gives terms that are always $\geqslant 1$.

The problem was enlarged by introducing two parameters, p and q, defining: