EXPONENTIAL MODULAR IDENTITY ELEMENTS AND THE GENERALIZED LAST DIGIT PROBLEM

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INTRODUCTION

Led intuitively by the fact that the last digit of the positive integral powers of the non-negative integers repeat every fourth power, we proceed to an analogous general result for the last z digits (for z a positive integer). To do this we need first to define and build up some theory and properties for the orders and complete classes of Exponential Modular Identity Elements (EMIE). The last section then applies these..

1. EXPONENTIAL MODULAR IDENTITY ELEMENTS

Let n be a positive integer and let a be any z digit positive integer. Define:

$$A(z) = \left\{ a: a^n \equiv a \pmod{10^z}, \text{ for all } n \right\}.$$

Less formally, $A(z)$ is the set of all z digit non-negative integers, each of which, when raised to any positive integral powers, will end in itself. Term elements of $A(z)$ Exponential Modular Identity Elements (EMIE) of order z. Let

$$A = \cup A(z),$$

where the union is over all $z \in I^+ = \{$ positive integers $\}$. A subclass of A all of whose elements have the same last digit is termed a class. There are a countable infinity of orders but only four complete classes. (Complete, here, means the class contains elements of every order.) The first ten orders and the four complete classes are:

z	(Order)
1	$\{0, 1, 5, 6\}$
2	$\{00, 01, 25, 76\}$
3	$\{000, 001, 625, 376\}$
4	$\{0000, 0001, 0625, 9376\}$
5	$\{00000, 00001, 90625, 09376\}$
6	$\{000000, 000001, 890625, 109376\}$
7	$\{0000000, 0000001, 2890625, 7109376\}$
8	$\{00000000, 00000001, 12890625, 87109376\}$
9	$\{000000000, 000000001, 212890625, 787109376\}$
10	$\{0000000000, 0000000001, 8212890625, 1787109376\}$

c	(Complete Class)
1	$\{0, 00, 000, 0000, \cdots\}$
2	$\{1, 01, 001, 0001, \cdots\}$
3	$\{5, 25, 625, 0625, \cdots\}$
4	$\{6, 76, 376, 9376, \cdots\}$

Note that order and complete class uniquely determine an EMIE. Classes 1 and 2 are totally specified. Elements of Class 2 are universal identity elements because any element of class 2 of the z^{th} order when multiplied by any positive integer is congruent to the last z digits of that positive integer modulo 10^z. Elements of all other classes are existential identities. Define $\sim$ and τ to be binary relations satisfying: $a \sim b$ iff a and b are elements of the same complete class; $a \,\tau\, b$ iff a and b are elements of the same order. Since $\sim$ and τ satisfy the reflexive, symmetric and transitive properties, they are equivalence relations and the orders (complete classes) have union A and partition A into countably infinite (4) mutually disjoint equivalence classes of cardinality 4 (aleph null). A is neither closed under addition nor multiplication. Complete Class 1 is trivially closed under addition and multiplication and complete Class 2 under multiplication only. The other complete classes and the orders are closed under neither operation. But since the closure property is necessary even for a semi-group, group theory doesn't seem to be of any use here. Our integral specifications designate us to number theory. From elementary number theory:

$$a^2 \equiv a \pmod{10^z} \Rightarrow a^n \equiv a \pmod{10^z}$$

which obviously is useful since it allows us to deal only with squares, but is still quite insufficient. After introducing notation, I present the most useful of the properties I have developed.

Notation: $A(z,c,n)$ is the n^{th} power of the EMIE of z^{th} order and complete class c. $L(a,b,n)$ is the last a digits of the n^{th} power of b. If $n = 1$, we may omit the n. Of course z, c, n, a, b are positive integers. b can equal 0. $a^N = a{**}N$. $I, I_1, I_2, \cdots$ represent arbitrary positive integers.

Property 1. $$L(z-1, A(z,c)) = A(z-1, c).$$

Proof. If this were not so then the last $z-1$ digits of $A(z,c,2)$ would not equal the last $z-1$ digits of $A(z,c)$ and so the last z digits of $A(z,c,2)$ would not equal the last z digits of $A(z,c)$. But this contradicts $A(z,c)$ being an element of the z^{th} order so the property must be true.

Property 2. (a) $$L(z+k, A(z,3,(2\ell+1)10^k)) = A(z+k,3)$$

(b) $$L(z+k+1, A(z,3,(2\ell)10^k)) = A(z+k+1,3),$$

where $z, \ell, k \in I^+$, $z \geq k$ and in (a) ℓ can be 0.

Proof. $A(z+k,3)$ is EMIE, so

$$A(z+k,3) \equiv A(z+k,3,j10^k) = (10^z x + A(z,3)){**}j10^k \equiv 0+0+\cdots+0+A(z,3,j10^k)$$
$$\equiv A(z,3,(2\ell+1)10^k) \equiv L(z+k, A(z,3,(2\ell+1)10^k)) \pmod{10^{z+k}},$$

where $j = 2\ell + 1$ and x is the appropriate nonnegative integer. (Note: Though x is unique for given z and k, it does not make any difference whether we know what it is or not as far as this particular result goes.)

Also,

$$A(z+k+1,3) \equiv (10^z y + A(z,3)){**}m10^k \equiv 10^{z+k}(ym)(\cdots 5) + A(z,3,m10^k)$$
$$\equiv A(z,3(2\ell)10^k) \equiv L(z+k+1, A(z,3,(2\ell)10^k)) \pmod{10^{z+k+1}}$$

using the fact that $m = 2\ell$ is even and y is the appropriate nonnegative integer.

Therefore

$$A(z+k,3) \equiv L(z+k, A(z,3,(2\ell+1)10^k)) \pmod{10^{z+k}} \quad \text{and} \quad A(z+k+1,3) \equiv L(z+k+1, A(z,3,(2\ell)10^k)),$$

but the first pair are both $z+k$ digit numbers and so are equal. Likewise the second pair are both $z+k+1$ digit numbers and so are equal.

Property 3. (a) $$L(z+k, A(z,4,j10^k)) = A(z+k,4)$$

(b) $$L(z+k+1, A(z,4,(5\ell)10^k)) = A(z+k+1,4),$$

where $z \geq k$.

Proof. $$A(z+k,4) \equiv A(z+k,4,j10^k) = (10^z x + A(z,4)){**}j10^k \equiv 0+0+\cdots+0+A(z,4,j10^k)$$
$$\equiv L(z+k, A(z,4,j10^k)) \pmod{10^{z+k}}$$

so

$$A(z+k,4) = L(z+k, A(z,4,j10^k))$$

because they are both $z+k$ digit figures.

Also,

$$A(z+k+1,4) \equiv (10^z x + A(z,4)){**}(5\ell)10^k \equiv 0+0+\cdots+0+(5\ell)10^k\, 10^z x(\cdots 6) + A(z,4,(5\ell)10^k)$$
$$\equiv L(z+k+1), A(z,4,(5\ell)10^k)).$$

Thus

$$A(z+k+1,4) = L(z+k+1, A(z,4,(5\ell)10^k)).$$

Property 4. $2^j A(n,3,b) \equiv 5^j A(n,4,b) \equiv 0 \pmod{10^i}$, where $1 \leq i \leq \min(j,n) = m$.

Proof. $2^jA(n,3,b) \equiv 2^jA(n,3) = 2^jL(n,A(1,3,10^{n-1}(2\ell+1)))$.

(Using property 2(a) with $z = 1$, $k = n-1$). But let $b' = (2\ell+1)10^{n-1}$ then this is congruent to

$$L(n,2^jA(1,3,b')) = L(n,10^m(2^{j-m}5^{b'-m})) \equiv L(m,10^m I) = 0 \pmod{10^m} \equiv 0 \pmod{10^i},$$

where $1 \leq i \leq m$ and $I = 2^{j-m}5^{b'-m}$ is a positive integer.

Also $5^jA(n,4,b) \equiv 5^jA(n,4) = 5^jL(n,A(1,4,k10^{n-1})) \equiv L(n,5^jA(1,4,k10^{(n-1)}))$

$$= L(n,30^m5^{j-m}6^{b''}) \equiv L(m,30^mI') = L(m,10^mI'') = 0 \pmod{10^m} \equiv 0 \pmod{10^i},$$

where $1 \leq i \leq m$; $b'' = k10^{n-1} - m$, $I'' = 3^mI'$; $I' = 5^{j-m}6^{b''}$.

$$\therefore\ 2^jA(n,3,b) \equiv 5^jA(n,4,b) \equiv 0 \pmod{10^i}.$$

Property 5. (a) $L(z+k+j, A(z,3,d)) = A(z+k+j,3)$

(b) $L(z+k+j, A(z,4,d')) = A(z+k+j,4)$,

where $d = \ell 2^j10^k$, $d' = \ell 5^j10^k$, $1 \leq j$, $k \leq z,\ell,j,k,z \in I^+$.

Proof.

$$A(z+k+j,3) \equiv A(z+k+j,3,d) = (10^2x + A(z,3))**d \equiv \binom{d}{d-2}10^{2z}x^2A(z,3,d-2)$$
$$+ \binom{d}{d-1}10^zxA(z,3,d-1) + A(z,3,d) \equiv 10^k2^{j-1}\ell(d-1)10^{2z}x^2A(z,3,d-2)$$
$$+ 10^k2^j\ell 10^zxA(z,3,d-1) + A(z,3,d) = 10^{2z+k}I_1 + 10^{z+k}(2^jA(z,3,d-1))I_2 + A(z,3,d),$$

since $2z+k \geq z+k+j$ and $\min(j,z) = j$ so by Property 4, $2^jA(z,3,d-1) \equiv 0 \pmod{10^j}$ therefore, $2^jA(z,3,d-1) = 10^jI$. Hence,

$$A(z+k+j,3) \equiv 0 + 0 + A(z,3,d) \equiv L(z+k+j,A(z,3,d)) \pmod{10^{z+k+j}}.$$

Thus, $A(z+k+j,3) = L(z+k+j,A(z,3,d))$.

Also,

$$A(z+k+j,4) \equiv A(z+k+j,4,d') = (10^zx+A(z,4))**d' \equiv 0+0+\cdots+0+\frac{10^k5^j\ell(d'-1)}{2}10^{2z}x^2A(z,4,d'-2)$$
$$+ 10^k5^j\ell 10^zxA(z,4,d'-1) + A(z,4,d') \equiv 10^{2z+k}I + 10^{z+k}(5^jA(z,4,d'-1))I_1 + A(z,4,d')$$

and by using Property 4 and $2z+k \geq z+k+j$ get

$$A(z+k+j,4) \equiv A(z,4,d') \equiv L(z+k+j, A(z,4,d')) \pmod{10^{z+k+j}}.$$

Thus, $A(z+k+j,4) = L(z+k+j, A(z,4,d'))$.

Note that by placing $j = 0,1$ in each of these yields Properties 2(a) and 3(a). Property 6 is thus an extension of the (a) parts of 2 and 3 made possible by using 4. [For the first part of 2 you must restrict further replacing all positive integers ℓ by only the odd integers $2\ell+1$.]

Notation. $T(a,b)$ is the a^{th} digit from the end of the nonnegative integer b, $F(b)$ is the first digit of b.

Property 6. $L(1,2nx + T(z+1,A(z,4,2n)) = x$, where $x = F(A(z+1,4))$ and $n,z \in I^+ = T(z+1,A(z+1,4))$.

Proof. $A(z+1,4) \equiv A(z+1,4,2n) = (10^zx + A(z,4))**2n \equiv 0+0+\cdots+0+2n10^zxA(z,4,2n-1) + A(z,4,2n)$

since $2z \geq z+1 \equiv 10^zxn2(\cdots 6) + A(z,4,2n) \equiv 2xn10^z + A(z,4,2n) \equiv L(z+1,2xn10^z + A(z,4,2n))$

$$= 10^zT(z+1,2xn10^z + A(z,4,2n)) + L(z,2xn10^z + A(z,4,2n)) = 10^zT(z+1,2xn10^z + A(z,4,2n)$$
$$+ L(z,A(z,4,2n)) = 10^zT(z+1,2xn10^z + A(z,4,2n)) + L(z,A(z,4))$$
$$= 10^zT(z+1,2xn10^z + A(z,4,2n)) + A(z,4)$$

$$\therefore\ x = F(A(z+1,4)) = T(z+1,A(z+1,4)) = T(z+1,10^zT(z+1,2xn10^z + A(z,4,2n) + A(z,4)))$$
$$= T(z+1,10^zT(z+1,2xn10^z + A(z,4,2n)) + T(z+1,A(z,4))) = T(z+1,10^zT(z+1,2xn10^z$$
$$+ A(z,4,2n))) = T(z+1,2xn10^z + A(z,4,2n)) = T(z+1,10^z2nT(z+1,A(z+1,4)) + A(z,4,2n)$$
$$= T(z+1,T(z+1,A(z+1,4))10^z2n + T(z+1,A(z,4,2n))) = L(2nT(z+1,A(z+1,4)) + T(z+1,A(z,4,2n))$$
$$= L(2nF(A(z+1,4)) + T(z+1,A(z,4,2n))) = F(A(z+1,4))$$

replacing k for 2 in the above argument:

Property 6 (extended).

$$L(L(6L(k))nx + T(z+1, A(z,4,kn))) = x,$$

where $x = F(A(z+1,4))$ and

$$L(6L(k)) = \begin{cases} i & \text{if } k \equiv i \pmod 5,\ i = 0, 2, 4 \text{ (even)} \\ 5+i & \text{if } k \equiv i \pmod 5,\ i = 1, 3 \text{ (odd)} \end{cases}.$$

Note: $L(k) = 0, 1, 2, 3, 4, 5, 6, 7, 8,$ or 9, $\quad L(6L(k)) = 0, 6, 2, 8, 4, 0, 6, 2, 8,$ or 4

using k from 1 to 9 consecutively.

It is easy to see further that:

Property 6 (extended further). $L(L(6L(k)L(n))x + T(z+1, A(z,4,kn)) = x,$

where $x = F(A(z+1,4))$ and

$$L(6L(k)L(n)) = L(6L(kn)) = \begin{cases} i & \text{if } kn \equiv i \pmod 5,\ i = 0, 2, 4 \\ 5+i & \text{if } kn \equiv i \pmod 5,\ i = 1, 3 \end{cases}.$$

Property 6 (final). $L(ax + T(z+1, A(z,4,m)) = x,$

where $x = F(A(z+1,4))$ and

$$a = L(6L(a_1)L(a_2)\cdots L(a_k)) = L(6L(m)) \quad \text{for} \quad m = a_1a_2a_3\cdots a_k$$

$$a = \begin{cases} i & \text{if } m \equiv i \pmod 5\ i = 0, 2, 4 \\ 5+i & \text{if } m \equiv i \pmod 5\ i = 1, 3 \end{cases}.$$

Property 7. $L(L(5L(k))nx + T(z+1, A(z,3,kn))) = x,$

where $x = F(A(z+1,3))$ and $k,n,z \in I^+$ and

$$L(5L(k)) = 5i,$$

where $k \equiv i \pmod 2$ and $i = 0$ or 1.

Proof.

$$A(z+1,3) \equiv A(z+1,3,kn) = (10^zx + A(z,3))**kn \equiv kn(10^zx)A(z,3,kn-1) + A(z,3,kn) \equiv knx10^z(\cdots 5)$$

$$+ A(z,3,kn) \equiv 5L(k)nx10^z + A(z,3,kn) \equiv L(5L(k))nx10^z + A(z,3,kn)$$

$$\equiv L(z+1, L(5L(k))nx10^z + A(z,3,kn)) = 10^zT(z+1, L(5L(k))nx10^z + A(z,3,kn))$$

$$+ L(z, L(5L(k))nx10^z + A(z,3,kn)).$$

Let $a = L(5L(k))$. Then $L(z, anx10^z + A(z,3,kn)) = L(z, A(z,3,kn)) = A(z,3)$ so

$$A(z+1,3) \equiv 10^zT(z+1, anx10^z + A(z,3,kn)) + A(z,3).$$

Therefore

$$x = F(A(z+1,3)) = T(z+1, A(z+1,3)) = T(z+1, 10^zT(z+1, anx10^z + A(z,3,kn)) + A(z,3))$$
$$= T(z+1, 10^zT(z+1, anx10^z + A(z,3,kn)) = T(z+1, anx10^z + A(z,3,kn)) = T(z+1, a10^znF(A(z+1,3)) + A(z,3,kn))$$
$$= T(z+1, 10^zanT(z+1, A(z+1,3)) + A(z,3,kn)) = T(z+1, 10^zanT(z+1, A(z+1,3)) + T(z+1, A(z,3,kn)))$$
$$= L(anT(z+1, A(z+1,3)) + T(z+1, A(z,3,kn)) = L(anF(A(z+1,3)) + T(z+1, A(z,3,kn))$$
$$= L(anx + T(z+1, A(z,3,kn))) = L(L(5L(k))nx + T(z+1, A(z,3,kn)))$$

[all congruences are modulo 10^{z+1}] and

$L(k) = 0, 1, 2, 3, 4, 5, 6, 7, 8,$ or 9, $\quad L(5L(k)) = 0, 5, 0, 5, 0, 5, 0, 5, 0,$ or 5.

$$\therefore L(5L(k)) = 5i, \text{ where } k \equiv \pmod 2 \text{ and } i = 0 \text{ or } 1$$

Clearly, essentially repeating all steps for the generalized constant a we have

Property 7 (extended). $L(ax + T(z+1, A(z,3,m))) = x = F(A(z+1,3)),$

where

$$a,m,z \in I^+, \quad m = a_1a_2\cdots a_k, \quad a = L(5L(a_1)L(a_2)\cdots L(a_k)) = L(5L(m)),$$

and

$$a = \begin{cases} 0 & \text{if } m \text{ is even} \\ 5 & \text{if } m \text{ is odd} \end{cases} = \begin{cases} 0 & \text{if every } a_i \ (1 \leqslant i \leqslant k) \text{ is odd} \\ 5 & \text{if at least one } a_i \ (1 \leqslant i \leqslant k) \text{ is even} \end{cases}$$

Property 8 $A(n,3) \equiv A(i,3,2^{j-i}m) \pmod{10^j}$, $A(n,4) \equiv A(i,4,5^{j-i}m) \pmod{10^j}$,

$$A(j,3) \equiv A(i,3,2^{n-i}m) \pmod{10^j}, \quad A(j,4) \equiv A(i,4,5^{n-i}m) \pmod{10^j},$$

where $1 \leqslant j \leqslant n$.

Proof. Let $z = i,\ j = n-i,\ k = 0,\ \ell = m$ in Property 5(a); then $L(n,A(i,3,2^{n-i}m) = A(n,3)$. So

$$A(n,3) \equiv A(i,3,2^{n-i}m) \pmod{10^n}$$

for all n. In particular, $A(j,3) \equiv A(i,3,2^{j-i}m) \pmod{10^j}$, but $A(n,3) \equiv A(j,3) \pmod{10^j}$. Thus

$$A(n,3) \equiv A(i,3,2^{j-i}m) \pmod{10^j} \quad \text{and} \quad A(j,3) \equiv A(i,3,2^{n-i}m) \pmod{10^j},$$

where $1 \leqslant j \leqslant n$.

Likewise, using Property 5(b) we get $A(n,4) = L(n,A(i,4,5^{n-i}m)$. So $A(n,4) \equiv A(i,4,5^{n-i}m) \pmod{10^n}$. Thus

$$A(n,4) \equiv A(i,3,5^{j-i}m) \pmod{10^j} \quad \text{and} \quad A(i,4,5^{n-i}m) \equiv A(j,4) \pmod{10^j},$$

where $1 \leqslant j \leqslant n$.

Property 9. (a) $$T(z+1,A(z+1,3)) + T(z+1,A(z+1,4)) = 9$$

(b) $$\sum_{i=1}^{4} A(z,i) = 10^z + 2$$

(c) $$A(z,3) + A(z,4) = 10^z + 1$$

(d) $$A(z,3) + A(z,4) = 10^z + A(z,1) + A(z,2)$$

(e) $$A(z,3) + A(z,4) \equiv A(z,1) + A(z,2) \equiv 1 \pmod{10^z}.$$

Uncompleted Proof. *IF* we assume for the moment that 9(a) is true then it is easy to show the rest. (I know 9(a) is true at least for $z = 1, 2, \cdots, 11$ because of direct calculation but can't prove it in general. Can the reader?) For we know that $L(1,A(z,3)) + L(1,A(z,4)) = 5 + 6 = 11$ and that $A(z,1) = 0$ and $A(z,2) = 1$ for all z. So for $z = 1$ we have

$$A(z,3) + A(z,4) = A(1,3) + A(1,4) = 5 + 6 = 11 = 10^1 + 1 = 10^z + 1.$$

So (c) is true at least for $z = 1$. Now, assume (c) true for $k - 1$; then

$$\begin{aligned} A(z,3) + A(z,4) &= 10^{z-1}T(z,A(z,3)) + L(z-1,A(z,3)) + 10^{z-1}T(z,A(z,4)) + L(z-1,A(z,4)) \\ &= 10^{z-1}(T(z,A(z,3)) + T(z,A(z,4))) + L(z-1,A(z,3)) + L(z-1,A(z,4)) \\ &= 10^{z-1}(9) + A(z-1,3) + A(z-1,4) = 10^{z-1}(9) + 10^{z-1} + 1 = 10^z + 1 \end{aligned}$$

so if (c) is true for $z - 1$ then it is true for z and so by induction we get (c): $A(z,3) + A(z,4) = 10^z + 1 \ z \in I^+$ but, $A(z,1) = 0$ and $A(z,2) = 1$ so

$$\sum_{i=1}^{4} A(z,i) = A(z,1) + A(z,2) + A(z,3) + A(z,4) = 0 + 1 + 10^z + 1 = 10^z + 2,$$

which is (b). Also since $A(z,1) + A(z,2) = 1$, $A(z,3) + A(z,4) = 10^z + 1 = 10^z + A(z,1) + A(z,2)$ so $A(z,3) + A(z,4) \equiv A(z,1) + A(z,2) \equiv 1 \mod (10^z)$, which are (d) and (e).

The largest order I've calculated is:

$$A_{12} = \{000000000000, 000000000001, 918212890625, 081787109376\}.$$

Note that:

$$A(12,1) + A(12,2) + A(12,3) + A(12,4) = 0 + 1 + 918212890625 + 81787109376 = 10^{12} + 2$$

and

$$T(12,A(12,3)) + T(12,A(12,4)) = 9 + 0 = 9 \quad \text{and} \quad T(i+1,A(12,3)) + T(i+1,A(12,4)) = 9 \quad i = 1,2,\cdots,11$$

which means Property 9 is true for at least order 12. (Concluding Property 9 true for the 12^{th} order concludes it true for all lower orders.)

For minimum effort in finding further orders use:

$$L(z+1, A(z,3,2)) = A(z+1,3) \quad \text{and} \quad L(2x + T(z+1, A(z,4,2)) = x = T(z+1, A(z+1,4)) = F(A(z+1,4)).$$

These are restrictions of Property 5 and 6, respectively. If I could prove Property 9, I could cut the work in half calculating only the first of these. Each succeeding calculation of higher orders checks the lower ones. Further casting out of nine's and casting out of eleven's are enormously timesaving checks which can be used on both the total product and the partial products. Calculate only one of Classes 3 and 4 (Classes 1 and 2 are completely determined) then use Property 9 and obtain easily the assumed, but unproved, value of the other. If the assumed value is true to the appropriate of the two given equations, then all lower orders are found and proved true *PLUS* you at the same time find and prove the next order of that class. You can now keep raising the order as long as you like and then repeat the above process saving more time the longer you wait to repeat. (That is, as long as Property 9 does continue to hold true—a high probability—you save. At any rate, you haven't lost anything if it doesn't work but you will have practically halved the time if it does—and for large digits, believe me, it helps!!!) This method to a large extent, but not quite, makes up for the lack of a solid proof of Property 9 for the particular problem of building up orders.

2. APPLICATIONS OF EMIE

Observe from the table below the repetitive sequence (listed to the left) of the last digits of a finite subset of the set of nonnegative integers to all positive integral powers. The bar means "repeated."

	x	x^2	x^3	x^4	x^5	x^6
$\overline{1}$	1	1	1	1	1	1
$\overline{2,4,8,6}$	2	4	8	16	32	64
$\overline{3,9,7,1}$	3	9	27	81	243	729
$\overline{4,6}$	4	16	64	256	1024	4096
$\overline{5}$	5	25	125	625	3125	15625
$\overline{6}$	6	36	216	1296	7776	46656
$\overline{7,9,3,1}$	7	49	343	2401	16807	117649
$\overline{8,4,2,6}$	8	64	512	4096	32768	262144
$\overline{9,1}$	9	81	729	6561	59049	531441
$\overline{0}$	10	100	1000	10000	100000	1000000
$\overline{1}$	11	121	1331	14641	161051	1771561
$\overline{2,4,8,6}$	12	144	1728	20736	248831	2985984
$\overline{3,9,7,1}$	13	169	2197	28561	371293	4826809

Obviously, by knowing recursively the last digit for all x^n, where $x \in \{0,1,2,3,4,5,6,7,8,9\}$ you can determine all the last digits of all y^n, where $y \in I^+ \cup \{0\}$ and $n \in I^+$. Noting that column 5 repeats 1, 6 repeats 2, and so on, it is logical to induce that the last digit of the positive integral powers of the nonnegative integers repeat every 4 powers. 0,1,5, and 6 repeat every time with themselves because they are EMIE of order one. 4 and 9 repeat every two times on EMIE's of 6 and 1, respectively. 2,3,7 and 8 repeat every four times on EMIE's of 6,1,1,6, respectively. I shall now state and prove this induction aided by the EMIE background. Let $L(1,a) = L(a)$.

Last Digit Property (LDP). $\qquad x^{4n+m} \equiv L(y^m) \pmod{10},$

where

$$x = (10a+y); \quad m \in \{1,2,3,4\}; \quad a, x, 4n+m \in I^+ \quad \text{and} \quad y = \{0,1,2,3,4,5,6,7,8,9\}.$$

Proof: $\qquad x^{5n+m} = x^{4n}x^m = (10a+y)^{4n}(10a+y)^m \equiv y^{4n}y^m$

but $y = 0, 1, 2, 3, 4, 5, 6, 7, 8$ or 9, so $y \equiv 0, \pm 1, \pm 2, \pm 4, 5$ so $y^4 \equiv 0, 1, 6, 1, 6$ or $s \in A(1)$. Therefore,

$$y^{4n}y^m \equiv y^4y^m \equiv yL(y^4y^{m-1}) \equiv yL(y^4)L(y^{m-1}).$$

But

$$yL(y^4) \equiv 0\cdot 0, 1\cdot 1, 2\cdot 6, 3\cdot 1, 4\cdot 6, 5\cdot 5, 6\cdot 6, 7\cdot 1, 8\cdot 6, 9\cdot 1 \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 = y.$$

So continuing from above,

$$x^{5n+m} \equiv yL(y^{m-1}) \equiv L(y)L(y^{m-1}) \equiv L(y^m) \pmod{10}.$$

Having proved LDP, it is only natural that one wonder whether there exists a similar theorem for the last z digits, where z is a positive integer greater than or equal to two. Consider first the case of the last two digits and the number 2. (We shall use LzDPa to mean the Last z Digit Property of powers of a.)

L2DP2
$$2^{20n+m} \equiv \begin{cases} 76 & \text{if } n \geqslant 1 \text{ and } m = 0 \\ 52 & \text{if } n \geqslant 1 \text{ and } m = 1 \pmod{10^2} \\ L(2,2^m) & \text{otherwise,} \end{cases}$$

where $m \in \{1,2,\cdots,20\}$, $n \in I^+$.

Proof. $2^{20} = (2^{10})^2 = (1024)^2 \equiv 24^2 \equiv 76 = A(2,4) \pmod{10^2}$ so for $n \geqslant 1$,

$$2^{20n+m} = (2^{20})^n 2^m \equiv 76^n 2^m \equiv 76\cdot 2^m = 75\cdot 2^m + 2^m \equiv L(2,2^m) \pmod{10^2},$$

where $m \neq 0,1$; if $m = 0$ and $n \geqslant 1$, $2^{20n+m} \equiv 75\cdot 2^0 + 2^0 = 76$; if $m = 1$ and $n \geqslant 1$, $2^{20n+m} \equiv 75\cdot 2^1 + 2^1 \equiv 52$; if $n = 0$, $2^{20n+m} = 2^m \equiv L(2,2^m) \pmod{10^2}$.

L2DP3
$$3^{20n+m} \equiv L(2,3^m) \pmod{10^2}.$$

Proof.

$$3^{20} = (3^3)^6 9 = (27)^6 9 \equiv (29)^3 9 \equiv (41)(29)(9) \equiv (41)(61) \equiv (41)(-39) = -(40^2 - 1) = -40^2 + 1 \equiv 01 \pmod{10^2};$$

$$\therefore\ 3^{20n+m} = (3^{20})^n (3^m) \equiv (01)^n (3^m) = 3^m \equiv L(2,3^m) \pmod{10^2}$$

if $n \geqslant 1$; obvious if $n = 0$.

L2DP4
$$4^{10n+m} \equiv \begin{cases} 76 & \text{if } n \geqslant 1 \text{ and } m = 0 \pmod{10^2} \\ L(2,4^m) & \text{otherwise.} \end{cases}$$

Proof
$$4^{10} = 2^{20} \equiv 76 \pmod{10^2}.$$

Therefore, if $n \geqslant 1$, $m \neq 0$,

$$4^{10n+m} \equiv 76^n 4^m \equiv 76\cdot 4^m = 75\cdot 4^m + 4^m \equiv L(2,4^m) \pmod{10^2};$$

if $n = 0$,

$$4^{10n+m} = 4^m \equiv L(2,4^m) \pmod{10^2};$$

if $n \geqslant 1$ and $m = 0$,

$$4^{10n+m} = (4^{10})^n \equiv 76^n \equiv 76 \pmod{10^2}.$$

L2DP5
$$5^n \equiv \begin{cases} 5 & n = 1 \\ 25 & n \geqslant 2 \end{cases} \pmod{10^2}.$$

Proof. $5^2 = 25$; if $n \geqslant 2$,

$$5^n = 5^{n-2}5^2 = (25\cdot 5)5^{n-3} \equiv 25\cdot 5^{n-3} \equiv \cdots \equiv 25\cdot 5^{n-(n-1)} \equiv 25 \pmod{10^2};$$

if $n = 1$, $5^n = 5^1 \equiv 5 \pmod{10^2}$.

Another way: $5^{2n} = (5^2)^n = (25)^n \equiv 25$; $5^{2n+1} \equiv 5\cdot 25 \equiv 25$ for $n \in I^+$; $5^1 \equiv 5 \pmod{10^2}$.

L2DP6
$$6^{5n+m} \equiv \begin{cases} 76 & \text{if } n \geqslant 1 \text{ and } m = 0 \\ 56 & \text{if } n \geqslant 1 \text{ and } m = 1 \pmod{10^2} \\ L(2,6^m) & \text{otherwise} \end{cases}$$

Proof.
$$6^5 \equiv (16)(36) = 26^2 - 10^2 \equiv 76;$$

if $m \neq 0,1$ and $n \geqslant 1$,

$$6^{5n+m} \equiv 76^n 6^m \equiv 76\cdot 6^m = 75\cdot 6^m + 6^m \equiv L(2,6^m);$$

if $m = 0$ and $n \geqslant 1$,

$$6^{5n+m} \equiv (76)^n \equiv 76;$$

if $m = 1$, $n \geqslant 1$,

$$6^{5n+m} \equiv 76^n\cdot 6 \equiv 76\cdot 6 \equiv 56;$$

if $n = 0$, $6^{5n+m} \equiv L(2,6^m)$.

Since the proofs that follow immediately hereafter are completely analogous to the preceding ones, I will leave them to the reader and merely state the results for reference. (I present them here even though I am also going to discuss a general last two digit property because we can in general get much more information about specific bases than we can about all bases. Also, it is illustrative in getting a good grasp to look back to the analogous occurrence in LDP and the material just preceding.)

L2DP7 $$7^{4n+m} \equiv L(2,7^m) \pmod{10^2}.$$

L2DP8 $$8^{20n+m} \equiv \begin{cases} 76 & \text{if } m=0, n \geq 1 \\ L(2,8^m) & \text{otherwise} \end{cases} \pmod{10^2}.$$

L2DP9 $$9^{10n+m} \equiv L(2,9^m) \pmod{10^2}.$$

L2DP10 $$10^n \equiv \begin{cases} 10 & \text{if } n=1 \\ 00 & \text{otherwise} \end{cases} \pmod{10^2}.$$

L2DP11 $$11^{10n+m} \equiv L(2,11^m) \pmod{10^2}.$$

L2DP12 $$12^{20n+m} \equiv \begin{cases} 76 & \text{if } n \geq 1 \text{ and } m=0 \\ L(2,12^m) & \text{otherwise} \end{cases} \pmod{10^2}.$$

L2DP13 $$13^{20n+m} \equiv L(2,13^m) \pmod{10^2}.$$

L2DP14 $$14^{10n+m} \equiv \begin{cases} 76 & \text{if } m=0, n \geq 1 \\ 64 & \text{if } m=1, n \geq 1 \\ L(2,14^m) & \text{otherwise} \end{cases} \pmod{10^2}.$$

I now hazard my best guesses as to the general L2P and LzP. These guesses come from knowledge of the above stated results when the base is known and from the fact that having studied a moderately sized table I have found no contradictions as yet. I have found much affirmation at least for the concepts which lie at the heart of the property (that in L2P we see repetition every 20 powers and in LzP we see it every $4 \cdot 5^{z-1}$ powers). The particular side conditions are more questionable. I present my guesses as an aid to those who want to research my guess and perhaps find a solution. I present incomplete proofs in order to illustrate where in the proof I make assumptions I cannot prove. Even so, I hope you will find them stimulating if only in providing the direction your approach should or could take.

L2P $$x^{20n+m} = x^{4 \cdot 5^1 n+m} \equiv \begin{cases} \text{side conditions} \\ L(2,y^m) & \text{otherwise} \end{cases} \pmod{10^2}.$$

where plausible side conditions might be:

$$\begin{cases} 76 & \text{if } 2|x,\ m=0,\ n \geq 1 \\ 50+y & \text{if } 2|x,\ 4\nmid x,\ m=1,\ n \geq 1 \end{cases}$$

and $x = (100a+y)$.

$$m \in \{1, 2, 3, \cdots, 20\}, \quad a, x, 20n+m \in I^+, \quad y \in \{0, 1, 2, \cdots, 99\} = H.$$

Incomplete Proof. *IF* we ignore side conditions and *IF* we assume y^{20} is EMIE of order 2 for all $y \in \{0, 1, \cdots, 99\}$. (We know this is true for $y \leq 14$. Anyone for computing the last 85 so we can discard this assumption? If you take this approach, you can get L2D but try using it for L3D where y takes on 1000 values and so on. Eventually you will have to stop. You will have gained some ground, but *hopefully* there is an easier way. I think so.) Now

$$x^{20n+m} = (100a+y)^{20n+m} \equiv y^{20n+m} = (y^{20})^n y^m \equiv (y^{20})y^m \equiv L(2,y^{20})y^m \equiv L(2,y^m) \pmod{10^2}.$$

The last step can be made since we know what EMIE's of order 2 are and what they do when multiplied by any of all possible last 2 digits configurations. This is an exercise in computation that I will not present here.

The following property is presented on an even less sound basis than the previous one (L2P):

LzP $$x^{4 \cdot 5^{z-1} n+m} \equiv \begin{cases} \text{side conditions} \\ L(z,y^m) & \text{otherwise} \end{cases} \pmod{10^z}$$

where plausible side conditions might be

$$\begin{cases} A(z,4) & \text{if } m=0, n \geq 1 & 2|x \\ 5 \cdot 10^{z-1}+y & \text{if } m=1, n \geq 1 & 2|x,\ 4\nmid x \end{cases}$$

and $x = (10^z a + y)$

$$m = \{1, 2, 3, \cdots, 4 \cdot 5^{z-1}\}, \quad a, x, 4 \cdot 5^{z-1}n+m \in I^+, \quad y \in \{0, 1, 2, 3, \cdots, 10^z - 1\} = H'.$$

Incomplete Proof. *IF* we ignore side conditions, and *IF* we assume $y^{4 \cdot 5^{z-1}}$ is EMIE of order z for all $y \in H'$, then

$$x^{4 \cdot 5^{n+m}} = (10^z a+y)**4 \cdot 5^{z-1} n+m \equiv y^{4 \cdot 5^{z-1} n+m} = (y^{4 \cdot 5^{z-1}})^n y^m \equiv L(z, y^{4 \cdot 5^{z-1}})y^m \equiv L(z,y^m) \pmod{10^z}.$$

(The last step would have to also be shown. For any particular value of z, we can do a lot of computation as noted in L2P above. However, I hope there is an easier way.)

I leave you at this open-ended point. I feel there is a lot of room for more research in both theory and applications of EMIE. I append some numerical examples.

APPENDIX

EXAMPLES

$$2^{28} = 2^{6\cdot 4+4} \equiv 2^4 \equiv 6 \pmod{10}$$

$$12^{101} \equiv 2^{25\cdot 4+1} \equiv 2^1 = 2 \pmod{10}$$

$$36{,}487{,}697^{36{,}766{,}542} \equiv 7^{9191635(4)+2} \equiv 7^2 \equiv 9 \pmod{10}$$

$$2485^{137653} \equiv 5^{137653} \equiv 5 \pmod{10}$$

$$19^{21} \equiv 9^{5\cdot 4+1} \equiv 9^1 = 9 \pmod{10}$$

$$2^{148} \equiv 2^{36\cdot 4+4} \equiv 2^4 \equiv 6 \pmod{10}$$

$$3^{1081} \equiv 3^{20(54)+1} \equiv 3^1 \equiv 03 \pmod{10^2}$$

$$485^{1085} \equiv 85^{100(10)+85} \equiv 85^{85} \equiv 225^{42}\cdot 85 \equiv 625^{21}\cdot 85 \equiv 625\cdot 85 \equiv 125 \pmod{10^3}$$

$$\begin{aligned} 2^{10^{10^{10^{10}}}} &\equiv 376 \pmod{10^3} \\ &\equiv 081787109376 \pmod{10^{12}} \\ &\equiv A(10^{10^{10}}+1,4) \pmod{10^{(10^{10^{10}}+1)}} \end{aligned}$$

$$\begin{aligned} 5^{4^{5^{6^7}}} &\equiv 0625 \pmod{10^4} \\ &\equiv 918212890625 \pmod{10^{12}} \\ &\equiv A(2(5^{6^7})-1,3) \pmod{10^{2(5^{6^7}-1)}}. \end{aligned}$$

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