A MAXIMUM VALUE FOR THE RANK OF APPARITION OF INTEGERS IN RECURSIVE SEQUENCES

H. J. A. SALLÉ
Laboratory of Medical Physics, University of Amsterdam, Herengracht 196, Holland

We define the sequence $R_0, R_1, R_2, \cdots$ by the recursive relation

$$R_{n+1} = aR_n + bR_{n-1}$$

in which $b = 1$ or -1; a and the discriminant $\Delta = a^2+4b$ are positive integers. In addition, we have the initial conditions $R_0 = 0$ and R_1 may be any positive integer. We now state the following:

Theorem. The rank of apparition of an integer M in the sequence $R_0, R_1, R_2, \cdots$ does not exceed $2M$.

Proof First we observe that R_1 divides all terms of the sequence. If the theorem holds for the sequence

$$0 = \frac{R_0}{R_1}, \quad 1 = \frac{R_1}{R_1}, \frac{R_2}{R_1}, \cdots$$

then it apparently holds for the sequence $R_0, R_1, R_2, \cdots$. Therefore we may suppose in what follows, that $R_1 = 1$.

Let M be a positive integer

$$M = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \quad .$$

Here $p_1, p_2, \cdots, p_k$ denote the different primes of M and $\alpha_1, \alpha_2, \cdots, \alpha_k$ their powers. To each p_i $(i = 1, 2, \cdots, k)$ we assign a number s_i:

$$s_i = p_i \pm 1 \text{ if } p_i \text{ is odd and } p_i \nmid \Delta;$$

the minus sign is to be taken if Δ is a quadratic residue of p_i and plus sign if it is a nonresidue

$$s_i = p_i \text{ if } p_i \text{ is odd and } p_i \mid \Delta.$$

$$s_i = 3 \text{ if } p_i = 2 \text{ and } \Delta \text{ odd.}$$

$$s_i = 2 \text{ if } p_i = 2 \text{ and } \Delta \text{ even}.$$

Let m be any common multiple of the numbers $s_1 p_1^{\alpha_1 - 1}, s_2 p_2^{\alpha_2 - 1}, \cdots, s_k p_k^{\alpha_k - 1}$ then $M \mid R_m$. In the case that m constitutes the least common multiple of the mentioned numbers, the proof can be found in Carmichael [1]. From the known property $R_q \mid R_{nq}$, n and q denote positive integers, it appears that m may be any common multiple (the property $R_q \mid R_{nq}$ can be found in Bachman [2]).

Now suppose that M contains only odd primes $p_1, p_2, \cdots, p_k$ with $p_1 \nmid \Delta, p_2 \nmid \Delta, \cdots, p_k \nmid \Delta$, then it is not difficult to verify that the product

$$m = 2\, \frac{s_1 p_1^{\alpha_1 - 1}}{2} \, \frac{s_2 p_2^{\alpha_2 - 1}}{2} \cdots \frac{s_k p_k^{\alpha_k - 1}}{2} \tag{1}$$

is a common multiple of the numbers $s_1 p_1^{\alpha_1 - 1}, \cdots, s_k p_k^{\alpha_k - 1}$ and therefore $M \mid R_m$. It is easy to verify that

$$\frac{m}{M} \leq \frac{4}{3} \quad .$$

The extension is easily made to the case where M contains also odd primes $q_1, q_2, \cdots, q_\ell$ with $q_1 \mid \Delta, \cdots, q_\ell \mid \Delta$ and /or to the case where M is even.

In the first case we form a common multiple by multiplying (1) with $q_1^{\beta_1} q_2^{\beta_2} \cdots q_\ell^{\beta_\ell}$ (the numbers $\beta_1, \cdots, \beta_\ell$ constitute the powers of $q_1, \cdots, q_\ell$ in M).

In the second case we multiply (1) with 2^γ if Δ is even and with $3.2^{\gamma-1}$ if Δ is odd (γ is the power of 2 which is contained in M). We now obtain

$$m \leqslant \tfrac{4}{3}M \text{ if } \Delta \text{ is even}$$
$$m \leqslant 2M \text{ if } \Delta \text{ is odd.}$$

This completes the proof.

SOME EXAMPLES

1. The Fibonacci sequence: $a = b = 1 \quad \Delta = 5 \quad R_1 = F_1 = 1$.
   If $M = 21$ then $p_1 = 3 \quad p_2 = 7$ so $s_1 = 4 \quad s_2 = 8$ and $m = 2\cdot\frac{4}{2}\cdot\frac{8}{2} = 16$.
   Therefore $21 | F_{16}$ (in fact $21|F_8$).
   If $M = 110 = 2\cdot 5\cdot 11$ then $m = 3\cdot 5\cdot 2\cdot\frac{10}{2} = 150$ so $110|F_{150}$.
   The only numbers having a rank of apparition equal to $2M$ are $6, 30, 150, 750, \cdots$ so $6|F_{12}$, $30|F_{60}$, $150|F_{300}$, etc.
2. The Pell numbers: $0, 1, 2, 5, 12, 29, 70, \cdots$ $a = 2 \quad b = 1 \quad \Delta = 8$.
   The numbers $3, 9, 27, \cdots$ constitute the only numbers having a rank of apparition equal to $\frac{4}{3}M$. So $3|R_4$, $9|R_{12}$, etc.

In the special case $b = -1$ the theorem can be strengthened. We use the same notation as before. First we prove the following

Lemma. Let $b = -1$. If p_i is an odd prime and $p_i \nmid \Delta$ then

$$p_i \big| R_{s_i/2}$$

Proof. We suppose again $R_1 = 1$. Next we introduce the auxiliary sequence $T_0, T_1, T_2, \cdots$ with $T_{n+1} = aT_n - T_{n-1}$ and the initial conditions $T_0 = 2 \quad T_1 = a$. The following properties apply: (Proof in Bachmann [2])

$$\text{I.} \quad p_i | R_{s_i}$$
$$\text{II.} \quad p_i | T_{s_i} - 2$$
$$\text{III.} \quad R_{2n} = R_n T_n \quad (n \text{ is a positive integer})$$
$$\text{IV.} \quad T_{2n} = T_n^2 - 2 \quad (n \text{ is a positive integer}).$$

Take $n = s_i/2$ in III and IV. From II and IV it follows

$$p_i \nmid T_{s_i/2}.$$

From I and III it then follows $p_i \big| R_{s_i/2}$. This proves the lemma. Now let M be again an integer

$$M = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}.$$

Further let m be the product of the numbers

$$(s_i p_i^{\alpha_i - 1})/2$$

respectively $s_i p_i^{\alpha_i-1}$ $(i = 1, 2, \cdots, k)$, where we have to choose the first number if p_i is an odd prime and $p_i \nmid \Delta$; the second number if $p_i | \Delta$ or $p_i = 2$. By Carmichael's method it can be proved that again $M | R_m$.

It is easy to verify that $m \leqslant M$ if Δ is even and that $m \leqslant \frac{3}{2}M$ if Δ is odd. So we have found:

The rank of apparition does not exceed M if $b = -1$ and Δ is even.
The rank of apparition does not exceed $\frac{3}{2}M$ if $b = -1$ and Δ is odd.

EXAMPLES

PREAMBLE: The equation $X^2 - NY^2 = 1$ in which N constitutes a positive integer, not a square, and X and Y are integers, is called Pell's equation. For given N, an infinite number of pairs X and Y exist, which satisfy the equation. If X_1 and Y_1 constitute the smallest positive solution, all solutions can be found from the recursive relations

$$X_{n+1} = 2X_1X_n - X_{n-1} \qquad Y_{n+1} = 2X_1Y_n - Y_{n-1}$$

with initial conditions $X_0 = 1$, $Y_0 = 0$.

The sequence $Y_0, Y_1, Y_2, \cdots$ does satisfy the conditions of the strengthened theorem.

EXAMPLE 1. Let $N = 3$, so $X^2 - 3Y^2 = 1$ then $X_1 = 2$, $Y_1 = 1$, $\Delta = 12$. The sequence $Y_0, Y_1, Y_2, \cdots$ consists of the numbers $0, 1, 4, 15, 56, 209, \cdots$. If $M = 110 = 2\cdot 5\cdot 11$ then $m = 2\cdot\frac{6}{2}\cdot\frac{10}{2} = 30$ so $110 \mid Y_{30}$. If $M = 18 = 2\cdot 3^2$ then $m = 2\cdot 3^2 = 18$ so $18 \mid Y_{18}$.

EXAMPLE 2. $X^2 - 2Y^2 = 1$ then $X_1 = 3,\ Y_1 = 2,\ \Delta = 32.$

The sequence $Y_0, Y_1, Y_2, \cdots$ consists of the numbers $0, 2, 12, 70, \cdots$ (which are Pell numbers with even subscript). The rank of apparition of any number M is less than M.

REMARK

If $b \neq \pm 1$ the theorem will generally not be valid; e.g., on taking $a = 4, b = 6, R_1 = 1$ any number M containing the factor 3 will not divide a member of the sequence.

REFERENCES

1. R.D. Carmichael, "On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$," *Annals of Mathematics*, Vol. 15, 1913, pp. 30–48.
2. P. Bachmann, "Niedere Zahlentheorie," 2er Teil, Leipzig, Teubner, 1910.

****

FIBONACCI AND LUCAS SUMS IN THE r-NOMIAL TRIANGLE

V.E. HOGGATT, JR., and JOHN W. PHILLIPS
San Jose State University, San Jose, California 95192

ABSTRACT

Closed-form expressions not involving $c_n(p,r)$ are derived for

(1) $$\sum_{n=0}^{p(r-1)} c_n(p,r) f_{bn+j}^m(x)$$

(2) $$\sum_{n=0}^{p(r-1)} c_n(p.r) \ell_{bn+j}^m(x)$$

(3) $$\sum_{n=0}^{p(r-1)} c_n(p,r)(-1)^n f_{bn+j}^m(x)$$

(4) $$\sum_{n=0}^{p(r-1)} c_n(p,r)(-1)^n \ell_{bn+j}^m(x),$$

where $c_n(p,r)$ is the coefficient of y^n in the expansion of the r-nomial

$$(1 + y + y^2 + \cdots + y^{r-1})^p, \qquad r = 2, 3, 4, \cdots, \qquad p = 0, 1, 2, \cdots,$$

and $f_n(x)$ and $\ell_n(x)$ are the Fibonacci and Lucas polynomials defined by

$$f_1(x) = 1, \qquad f_2(x) = x, \qquad f_n(x) = xf_{n-1}(x) + f_{n-2}(x);$$
$$\ell_1(x) = x, \qquad \ell_2(x) = x^2 + 2, \qquad \ell_n(x) = x\ell_{n-1}(x) + \ell_{n-2}(x).$$

Fifty-four identities are derived which solve the problem for all cases except when both b amd m are odd; some special cases are given for that last possible case. Since $f_n(1) = F_n$ and $\ell_n(1) = L_n$, the n^{th} Fibonacci and Lucas numbers respectively, all of the identities derived here automatically hold for Fibonacci and Lucas numbers. Also, $f_n(2) = P_n$, the n^{th} Pell number. These results may also be extended to apply to Chebychev polynomials of the first and second kinds.

The entire text of this 51-page paper is available for $2.50 by writing the Managing Editor, Brother Alfred Brousseau, St. Mary's College, Moraga, California 94575.

****