ELEMENTARY PROBLEMS AND SOLUTIONS

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-316 Proposed by J.A.H. Hunter, Fun with Figures, Toronto, Ont., Canada.

Solve the alphametic:

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  TWO
THREE
THREE
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EIGHT
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Believe it or not, there must be no 8 in this!

B-317 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that L_{2n-1} is an exact divisor of $L_{4n-1} - 1$ for $n = 1, 2, \cdots$.

B-318 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that $F_{4n}^2 + 8F_{2n}(F_{2n} + F_{6n})$ is a perfect square for $n = 1, 2, \cdots$.

B-319 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Prove or disprove:

$$\frac{1}{L_2} + \frac{1}{L_6} + \frac{1}{L_{10}} + \cdots = \frac{1}{\sqrt{5}}\left(\frac{1}{F_2} - \frac{1}{F_6} + \frac{1}{F_{10}} - \cdots\right) .$$

B-320 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m} .$$

B-321 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m+1}.$$

SOLUTIONS

A COMBINATORIAL PROBLEM

B-292 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain and prove a formula for the number $S(n,t)$ of terms in $(x_1 + x_2 + \cdots + x_n)^t$, where n and t are integers with $n > 0$, $t \geqslant 0$.

I. Solution by Graham Lord, Secane, Pennsylvania.

$S(n,t)$ is the number of unordered selections of size t and a set of n elements, that is:

$$S(n,t) = \binom{n+t-1}{t}.$$

This is a well known result. See for example H.H. Ryser, "Combinatorial Mathematics," *Carus Monograph,* American Math Association, 1963.

II. Solution by Frank Higgins, Naperville, Illinois.

$$S(n,t) = \binom{n+t-1}{t}.$$

For $n = 1$, the formula clearly holds for all integers $t \geqslant 0$. Suppose the formula holds for some integer $n \geqslant 1$ and all integers $t \geqslant 0$. Now, for any integer $t \geqslant 0$, we have that

$$(x_1 + x_2 + \cdots + x_n + x_{n+1})^t = [(x_1 + x_2 + \cdots + x_n) + x_{n+1}]^t = \sum_{k=0}^{t} \binom{t}{k} (x_1 + x_2 + \cdots + x_n)^{t-k} x_{n+1}^k$$

and hence, by the induction hypothesis, that

$$S(n+1, t) = \sum_{k=0}^{t} \binom{n+t-k-1}{t-k} = \binom{n+t}{t}$$

which completes the proof.

Also solved by Paul S. Bruckman, Jeffrey Shallit, A.C. Shannon, Gregory Wulczyn, and the Proposer.

THE FIRST SIX FIBONACCI TERMS

B-293 Proposed by Harold Don Allen, Nova Scotia Teachers College, N.S., Canada.

Identify T, W, H, R, E, F, I, V and G as distinct digits in $\{1, 2, \cdots, 9\}$ such that we have the following sum (in which 1 and 0 are the digits 1 and 0):

$$\begin{array}{r} 1 \\ 1 \\ TWO \\ THREE \\ \underline{FIVE} \\ EIGHT \end{array}$$

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

It is easy to see that the digit carried from the thousands column must be 1; consequently, $T+1=E$. Applying this fact to the ones column yields the congruence $2T+4 \equiv T \pmod{10}$ whose only solution is $T=6$. Therefore, $E=7$ follows. On the basis of the thousands column one can also easily deduce that $I \leqslant 5$. Furthermore, it is evident that the values of V and W are interchangeable. The value of H determines the possible values for V and W, resulting in the following ten cases:

(1) $H=1;\ V,W \in \{5,8\}$:	(6) $H=4;\ V,W \in \{1,5\}$:
(2) $H=1;\ V,W \in \{4,9\}$;	(7) $H=5;\ V,W \in \{3,4\}$;
(3) $H=2;\ V,W \in \{1,3\}$;	(8) $H=5;\ V,W \in \{8,9\}$;
(4) $H=2;\ V,W \in \{5,9\}$;	(9) $H=8;\ V,W \in \{1,9\}$;
(5) $H=3;\ V,W \in \{1,4\}$;	(10) $H=9;\ V,W \in \{3,8\}$.

All but two of these lead to contradictions. Case (4) yields one solution, from Case (9) two solutions are obtained; they are given below.

1	1	1
1	1	1
690	610	610
62477	68577	68277
8157	3297	5497
71326	72486	74386

As remarked earlier, upon interchanging the values of V and W, three additional solutions may be given. It may be of interest to note that the number of essentially different solutions, the possible values of F (commonly used to denote the Fibonacci numbers), as well as the possible values of H (often used to denote generalized Fibonacci numbers) are all Fibonacci numbers.

Also (partially) solved by Paul S. Bruckman, Warren Cheves, J.A.H. Hunter, John W. Milsom, Carl Moore, Jim Pope, A.C. Shannon, and the Proposer.

A FORMULA SYMMETRIC IN k AND n

B-294 Proposed by Richard Blazej, Queens Village, New York.

Show that

$$F_nL_k + F_kL_n = 2F_{n+k}.$$

Solution by Frank Higgins, Naperville, Illinois.

Using the Binet formulas we have

$$F_nL_k + F_kL_n = \left(\frac{a^n-b^n}{\sqrt{5}}\right)(a^k+b^k) + \left(\frac{a^k-b^k}{\sqrt{5}}\right)(a^n+b^n) = 2\left(\frac{a^{n+k}-b^{n+k}}{\sqrt{5}}\right) = 2F_{n+k}.$$

Also solved by George Berzsenyi, Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Mike Hoffman, Peter A. Lindstrom, Graham Lord, John W. Milsom, Carl Moore, F.D. Parker, Jeffrey Shallit, A.C. Shannon, Paul Smith, Gregory Wulczyn, and the Proposer.

CONVOLUTION OR DOUBLE SUM

B-295 Proposed by V.E. Hoggatt, Jr., California State University, San Jose, California.

Find a closed form for

$$\sum_{n=1}^{n} (n+1-k)F_{2k} = nF_2 + (n-1)F_4 + \cdots + F_{2n}.$$

Solution by Graham Lord, Secane, Pennsylvania.

The sum of the first k odd indexed Fibonacci numbers is F_{2k} and that of the first k even indexed ones is $F_{2k+1} - 1$, where $k \geqslant 1$.

Therefore,

$$\sum_{k=1}^{n} (n+1-k)F_{2k} = \sum_{j=1}^{n}\sum_{i=1}^{i} F_{2i} = \sum_{j=1}^{n} (F_{2j+1} - 1)$$
$$= F_{2(n+1)} - n - 1 .$$

NOTE: Compare B-290.

Also solved by George Berzsenyi, Paul S. Bruckman, Herta T. Freitag, Frank Higgins, Mike Hoffman, Peter A. Lindstrom, Carl Moore, Jeffrey Shallit, A.C. Shannon, Paul Smith, and the Proposer.

A MOST CHALLENGING PROBLEM

B-296 Proposed by Gary Ford, Vancouver, B.C., Canada.

Find constants a and b and a transcendental function G such that

$$G(y_{n+3}) + G(y_n) + G(y_{n+2})G(y_{n+1})$$

whenever y_n satisfies $y_{n+2} = ay_{n+1} + by_n$.

I. Solution by Carl F. Moore, Tacoma, Washington.

Two solutions are given by:

(1) $\quad a = b = 1 \quad$ and $\quad G(u) = 2\cos u,$

(2) $\quad a = b = 1 \quad$ and $\quad G(u) = c^u + c^{-u} \quad (c \neq 1).$

[Notice $G(u) = 2\cosh u$ is a pleasing special case.]

To show (1),

$$G(y_{n+3}) + G(y_n) = 2\cos(y_{n+3}) + 2\cos(y_n) = 2(\cos(y_{n+3}) + \cos(y_n))$$
$$= 2\left(2\cos\frac{y_{n+3}+y_n}{2}\cdot\cos\frac{y_{n+3}-y_n}{2}\right)$$
$$= 2\left(2\cos\frac{2y_{n+2}}{2}\cdot\cos\frac{2y_{n+1}}{2}\right)$$
$$= (2\cos(y_{n+2}))\cdot(2\cos(y_{n+1})) = G(y_{n+2})\cdot G(y_{n+1}).$$

To show (2),

$$G(y_{n+3}) + G(y_n) = (c^{y_{n+3}} + c^{-y_{n+3}}) + (c^{y_n} + c^{-y_n}) = c^{y_{n+2}+y_{n+1}} + c^{-y_{n+2}-y_{n+1}} + c^{y_{n+2}-y_{n+1}}$$
$$+ c^{y_{n+1}-y_{n+2}} = c^{y_{n+2}}\cdot c^{y_{n+1}} + c^{-y_{n+2}}\cdot c^{-y_{n+1}} + c^{y_{n+2}}\cdot c^{-y_{n+1}} + c^{y_{n+1}}\cdot c^{-y_{n+2}}$$
$$= (c^{y_{n+2}} + c^{-y_{n+2}})\cdot(c^{y_{n+1}} + c^{-y_{n+1}}) = G(y_{n+2})\cdot G(y_{n+1}).$$

II. Solution by the Proposer.

Let $G(x) = c^x + c^{-x}$, with c any (complex) constant and let $\{y_n\}$ be a generalized Fibonacci sequence (satisfying $y_{n+2} = y_{n+1} + y_n$ and having any initial conditions).

There were no other solvers.

PARTIAL FRACTIONS

B-297 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Obtain a recursion formula and a closed form in terms of Fibonacci and Lucas numbers for the sequence (G_n) defined by the generating function:

$$(1-3x-x^2+5x^3+x^4-x^5)^{-1} = G_0+G_1x+G_2x^2+\cdots+G_nx^n+\cdots.$$

Solution by David Zeitlin, Minneapolis, Minnesota.

We note that

$$G_{n+5}-3G_{n+4}-G_{n+3}+5G_{n+2}+G_{n+1}-G_n = 0.$$

Since

$$(1-3x-x^2+5x^3+x^4-x^5) = (1-3x+x^2)(1-x-x^2)(1+x),$$

we obtain, using partial fractions,

$$\frac{10}{1-3x-x^2+5x^3+x^4-x^5} = \frac{18-7x}{1-3x+x^2} - \frac{5(2+x)}{1-x-x^2} + \frac{2}{1+x}.$$

If $W_{n+2}=aW_{n+1}+bW_n$, then

$$\sum_{n=0}^{\infty} W_nx^n = \frac{W_0+(W_1-aW_0)x}{1-ax-bx^2}.$$

Thus,

$$\frac{18-7x}{1-3x+x^2} = \sum_{n=0}^{\infty} L_{2n+6}x^n; \qquad \frac{2+x}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+3}x^n; \qquad \frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^nx^n.$$

Thus,

$$G_n = \frac{1}{10}\left(L_{2n+6}-5F_{n+3}+2(-1)^n\right).$$

Also solved by Frank Higgins, Carl F. Moore, A.C. Shannon, Gregory Wulczyn, and the Proposer.

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