ON THE ORDER OF SYSTEMS OF TWO SIMULTANEOUS LINEAR DIFFERENCE EQUATIONS IN TWO VARIABLES

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1. INTRODUCTION

Several techniques are known for solving general linear difference equations [1, 2, 3]. We are not concerned here with specific techniques for actually solving difference equations (whether numerically or symbolically). Rather, the main problem dealt with is that of determining the *order* (number of initial conditions) of systems of two simultaneous linear difference equations in two variables. Since the order for non-homogeneous equations is that of the associated homogeneous equations, we content ourselves with the homogeneous case. This paper examines the definition of two-dimensional sequences by a system of two simultaneous linear difference equations in two variables and the initial value problem is solved algorithmically.

While it is relatively simple in the one-dimensional case to specify suitable initial conditions, the same problem in two dimensions is considerably more complicated. The traditional algebraic approach relies upon the representation of the elements of two-dimensional sequences as the matrix product of two geometric progressions, one considered as a row matrix, the other as a column matrix.

In the author's algorithmic approach a suitably defined finite subset of elements of the sequence is selected. Using constraints determined by the difference equations, certain elements of the subset are chosen whose values are determined by the values of the remaining elements of the subset that in turn are determined by initial values. Induction is used to prove that the entire sequence is determined by the initial values. The algorithm has been programmed in FORTRAN.

2. THE DEFINITION OF TWO-DIMENSIONAL FIBONACCI SEQUENCES

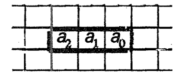
Any linear difference equation in one variable can be written in the following form:

$$c_0 f(i+m_0) = \sum_{k=1}^{n} c_k f(i+m_k) \ , \tag{1}$$

where f is a function on the integers, i.e., a sequence, $M = (m_k)$ is a vector with $n+1$ integer components, and the c's are non-zero coefficients and are distinct. For some purposes, it is more convenient to express linear difference equations diagrammatically rather than strictly algebraically as in (1). For example, the diagram, or *pattern* as we will call it, for the Fibonacci recursion relation is shown in Fig. 1. The two variable equation corresponding to (1) is

$$c_0 f(i+m_0, j+n_0) = \sum_{k=1}^{n} c_k f(i+m_k, j+n_k), \tag{2}$$

where f is a function of two integer variables, m_k corresponds to the column index for the k^{th} term, and n_k corresponds to the row index. $M = (m_k)$ and $N = (n_k)$ are vectors with $n+1$ integer components with $(m_i, n_i) \neq (m_j, n_j)$ if $i \neq j$. The c's are non-zero coefficients.



$$a_0 = a_1 + a_2$$

Figure 1 Pattern for the Fibonacci Recursion Relation

One of the simplest non-trivial two dimensional sequences is the "Fibonacci Multiplication Table," derived from the simultaneous equations

$$f(i+2, j) = f(i+1, j) + f(i, j) \tag{3}$$

$$f(i, j+2) = f(i, j+1) + f(i, j). \tag{4}$$

The pattern for Eq. (3) is

	a_2	a_1	a_0	

$$a_0 = a_1 + a_2$$

and the pattern for Eq. (4) is

	b_2	
	b_1	
	b_0	

$$b_0 = b_1 + b_2 \; .$$

Equations (3) and (4) with initial conditions

$$f(0, 0) = 0, \qquad f(0, 1) = 0,$$

$$f(1, 0) = 0, \qquad f(1, 1) = 1,$$

lead to a sequence with the property that

$$f(1, j) = f(i, 1) f(1, j). \tag{5}$$

Since row 1 and column 1 contain ordinary Fibonacci sequences, the sequence may be looked at as a multiplication table for the Fibonacci numbers.

3. INITIAL CONDITIONS FOR LINEAR DIFFERENCE EQUATIONS

In the one-dimensional case, the order is easily determined by inspection (see [4]). If the equation is written in the form of (1), the order, which we will call N_g is

$$N_g = \max_{i,j} |m_i - m_j| \, . \tag{6}$$

This number N_g is also one less than the width in grid squares of the pattern for the equation.

The set $G = \quad g_k$ giving a possibility for the relative positions of the initial values will be diagrammed on a grid in analogy to the way patterns are diagrammed. For example, the pattern

	a_2		a_1		a_0	

$$a_0 = a_1 + a_2$$

requires four initial values, since the width is 5. One valid g-pattern for these initial values is

	g_1		g_2	g_2		g_4

In the traditional algebraic approach to initial conditions in two dimensions, we represent the solution by a matrix product of two geometric progressions. If we use R for the horizontal ratio and S for the vertical ratio, then the analog of Eq. (2) is

$$c_0 R^{m_0} S^{n_0} = \sum_{k=1}^{n} c_k R^{m_k} S^{n_k} \, . \tag{7}$$

If we form the Eqs. (7) for two patterns, and let the first pattern have degree d_1 in R and e_1 in S, and the equation for the second pattern have degree d_2 in R and e_2 in S, then solving the equations simultaneously using the resultant (see [5]), we find there are at most M_g initial conditions required, where

$$M_g = (d_1 + d_2) \max(e_1, e_2) \min(d_1, d_2). \tag{8}$$

This method may require much tedious algebraic manipulation. Also, the theory does not provide in general even one valid g-pattern. The algorithm described in the following section solves the initial value problem without relying on geometric progressions. Also, it has the advantage of yielding a family of valid g-patterns.

4. AN EFFICIENT ALGORITHM FOR DETERMINING SETS OF INITIAL CONDITIONS IN TWO DIMENSIONS

Given two patterns for two linear difference equations in two unknowns, the algorithm described in this section first constructs a special set of adjacent grid squares (corresponding to the elements of a two-dimensional sequence) called a *starting set.* Then the number of initial conditions necessary and sufficient to determine the values for all of the elements in the starting set is calculated by matrix operations on the coefficients of equations implied by the difference equations. A form of two-dimensional induction is attempted to check whether the values for the entire sequence can be determined from the equations already solved and the values for the elements of the starting set. If the induction step fails, either the equations were not independent, or the starting set was not large enough. Assuming the latter, the starting set is enlarged and the procedure is repeated until either the induction step succeeds, or too many initial conditions are required for the equations to have been independent.

ALGORITHM FOR THE TWO-DIMENSIONAL INITIAL VALUE PROBLEM

Given two patterns with elements labelled $a_0, a_1, \cdots, a_n$ and $b_0, b_1, \cdots, b_m$, representing two linear difference equations in two variables, find N_g, the number of initial values necessary and sufficient to define a complete two-dimensional sequence and find at least one valid g-pattern if N_g is finite.

STEP 1. (Initialize the first starting set.) Fix the position of the pattern whose squares are labelled with a's on a grid. Let S be the set of all grid squares necessary and sufficient to represent

$$a_i = b_0 = \sum_{k=1}^{m} b_k$$

for $i = 0, 1, \cdots, n.$ (Note that the squares representing $a_0, a_1, \cdots, a_n$ are in S.)

STEP 2. (Check for horizontal gaps.) If there is no element of $\bar{S}$ between two elements of S in the same row of the grid, then go to Step 4.

STEP 3. (Augment S to reduce a horizontal gap.) Replace S by $S \cup R,$ where R is the set of all grid squares one grid square to the right of a grid square in $S.$ Go to Step 2.

STEP 4. (Check for vertical connectedness.) If each row containing an element of S (except the bottom-most) contains an element of S that is vertically adjacent to an element of S in the next row down, then go to Step 6.

STEP 5. (Augment S to reduce a vertical gap.) Replace S by $S \cup B,$ where B is the set of all grid squares one grid square below an element of $S.$ Go to Step 2.

STEP 6. (Set up equations.) Associate the i^{th} grid square in S with the variable x_i. Form M' as a coefficient matrix with columns representing the variables x_i and whose rows are the coefficients of all possible equations determined by the two patterns and involving only elements of the starting set $S.$

STEP 7. (Echelonize M'.) Put M' into echelon form $M.$

STEP 8. (Count free variables.) Label the distinguished column variables of M with x_i's, and the free-variable columns with g_i's. Let n_g = the number of g_i's. (Note that n_g is also the difference between the number of columns and the number of non-zero rows of $M.$)

STEP 9. (Check for dependent equations.) If $n_g > M_g$ from Eq. (8) then stop. $N_g = \infty$.

STEP 10. (Check horizontal induction.) Check whether M is row-equivalent to a matrix G, in echelon form, all of whose free variables correspond to grid squares of S that have a grid square corresponding to a distinguished variable on the right. (In forming $G,$ columns of M may be interchanged if a non-zero value appears in both columns for any one row.) If so, go to Step 12.

STEP 11. (Augment $S.$) Replace S by $S \cup T,$ where T is the set of all elements that are one grid square left of, right of, above, or below an element of $S.$ Go to Step 6.

STEP 12. (Check vertical induction.) Check whether M is row equivalent to an echelon matrix H all of whose free variables correspond to grid squares that have a grid square corresponding to a distinguished variable one grid square below. If not, go to Step 11. Otherwise, the algorithm terminates, N_g is equal to the n_g calculated in Step 8. The grid squares correspond to the g_k's for the matrices $M, G,$ or $H,$ form valid g-patterns. □

The goal of Steps 1 through 5 is to find a starting set with the following properties:

(1) The set should be connected, that is, it should be possible to go from each element to every other element remaining within the set and using only moves of one square horizontally or vertically.

(2) Every element of S should appear in at least one equation formed in Step 6.

The reason behind requirement (1) is that it has been found empirically that when it is satisfied the algorithm never needs to execute Step 11 and repeat Step 6 and the following steps. This has not been proved, however. The reason for requirement (2) is to avoid introducing extraneous free variables into the starting set. If an element appears in no equations for a particular starting set, that element will always appear to be a free variable, even though it would not necessarily be free if a larger starting set were used that allowed it to appear in an equation.

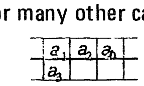
We now give a proof that the number n_g calculated in Step 8 is always a lower bound on the number of initial conditions necessary to define a complete sequence.

Proof. If Step 12 is reached and is successful, then n_g is a sufficient number of initial conditions, and all the values for elements of the sequence outside the starting set are derivable from the values of the starting set given n_g initial conditions in the positions of the free variables (the g's). Including equations involving elements outside the starting set would not add any new information to the system. If either the horizontal or the vertical induction fails, and all bordering values are not deriveable, then, since each g_i is necessary because at least one x_j depends on it, n_g is a lower bound on the number of initial conditions required. ☐

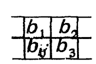
The procedure must terminate (i.e., is an algorithm) because, if a finite number of initial conditions exists, the starting set must eventually include at least one possible set of locations for those initial conditions, since all of the elements in the sequence are eventually included in the starting set.

The claim for efficiency in the title of this section is based on the observation that, for all cases tried, the number of zero rows in the matrix M, the echelon form of M', is $(N3+1)(N5+1)$, where $N3$ is the number of times Step 3 was executed in constructing S, and $N5$ is the number of times Step 5 was executed. This means that, if deriving values for the elements of a two-dimensional sequence is the object of discovering the number N_g and a valid g-pattern, most of the rows of M, with the exception of a limited number of zero rows, are useful for back-substitution in M given values for the g_i's. Also, using the two-dimensional induction technique, the value for any element in the sequence can be determined using only repeated back-substitution in M.

As a specific example of the algorithm, we give the results for the two patterns shown in Fig. 2. The number of initial conditions n_g for this case is 3, and a valid g-pattern is shown in Fig. 3. A portion of the two-dimensional sequence determined by $g_1 = 0$, $g_2 = 1$, $g_3 = 2$ is shown in Fig. 4. More detail on the operation of the algorithm, as well as the results for many other cases, are given in [6].



$$a_0 = a_1 + a_2 + a_3$$



$$b_0 = b_1 + b_2 + b_3$$

Fig. 2 Patterns for $f(m+2, n+1) = f(m, n+3) + f(m+1, n+1) + f(m,n)$ and $f(m,n) = f(m, n+3) + f(m+1, n+1) + f(m+1, n)$

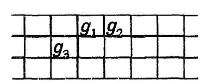


Fig. 3 A valid g-pattern for the patterns in Fig. 2

			43	−30	21	−14	11	-6	5	-6	-5
−23	16	−11	8	-5	4	-3	0	−5	−4	-3	8
−4	3	−2	1	−2	−1	−2	1	6	15	22	17
−1	0	−1	0	1	4	7	8	1	−20		
0	1	2	3	2	−3	−14	−29	−38	−19		
1	0	−3	-8	−13	−12	5	48				
−4	−5	−2	9	30	55						
7	16	23	16	−23							
0	−23										

Fig. 4 A portion of a two-dimensional sequence satisfying the patterns shown in Fig. 2. The initial values are circled.

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If we add any quantity B to each term, the above becomes

$$(x^2+B)^m - 3[(x+1)^2+B]^m + 3[(x+2)^2+B]^m - [3[(x+2)^2-(x+1)^2]+x^2+B]^m$$
$$= -B^m - 3[x^2-4(x+1)^2+3(x+2)^2+B]^m + 3[x^2-3(x+1)^2-4(x+2)^2+B]^m + [3[(x+2)^2-(x+1)^2]+B]^m$$

(where $m = 1, 2$).

Finally, take the series in which

$$F_n = A_{n-1}F_{n-1} + A_{n-2}F_{n-2} \cdots A_2F_2 + A_1F_1 .$$

We conjecture that

$$A_1F_1^m + A_2F_2^m + A_3F_3^m \cdots A_{n-2}F_{n-2}^m + A_{n-1}F_{n-1}^m + \left(\sum_1^{n-1} A - 2\right) F_n^m$$

(2)

$$= A_1(F_n - F_1)^m + A_2(F_n - F_2)^m + A_3(F_n - F_3)^m \cdots A_{n-2}(F_n - F_{n-2})^m + A_{n-1}(F_n - F_{n-1})^m + \left(\sum_1^{n-1} A - 2\right) 0^m$$

(where $m = 1, 2$).

Proof: When $m = 1$,

$$\text{L.H.S.} = \left(\sum_1^{n-1} A - 1\right) F_n .$$

When $m = 1$,

$$\text{R.H.S.} = (A_1 + A_2 + A_3 \cdots A_{n-2} + A_{n-1})F_n - (A_1F_1 + A_2F_2 + A_3F_3 \cdots A_{n-2}F_{n-2} + A_{n-1}F_{n-1}) = \left(\sum_1^{n-1} A - 1\right) F_n .$$

$$\therefore \text{ L.H.S.} = \text{R.H.S.}$$

When $m = 2$,

$$\text{L.H.S.} = A_1F_1^2 + A_2F_2^2 + A_3F_3^2 \cdots A_{n-2}F_{n-2}^2 + A_{n-1}F_{n-1}^2 + \left(\sum_1^{n-1} A - 2\right) F_n^2 .$$

When $m = 2$,

$$\begin{aligned}\text{R.H.S.} &= A_1F_n^2 - 2A_1F_1F_n + A_1F_1^2 + A_2F_n^2 - 2A_2F_2F_n + A_2F_2^2 \\ &\qquad + A_3F_n^2 - 2A_3F_3F_n + A_3F_3^2 + \cdots \\ &\qquad + A_{n-1}F_n^2 - 2A_{n-1}F_{n-1}F_n + A_{n-1}F_{n-1}^2 \\ &= \sum_1^{n-1} AF_n^2 - 2F_n \cdot F_n + A_1F_1^2 + A_2F_2^2 + A_3F_3^2 \cdots A_{n-1}F_{n-1}^2 \\ &= \left[\sum_1^{n-1} A - 2\right] F_n^2 + A_1F_1^2 + A_2F_2^2 + A_3F_3^2 \cdots A_{n-1}F_{n-1}^2 = \text{L.H.S.}\end{aligned}$$

If we add any quantity B to each term, we get

$$A_1(F_1+B)^m + A_2(F_2+B)^m + A_3(F_3+B)^m \cdots A_{n-2}(F_{n-2}+B)^m + A_{n-1}(F_{n-1}+B)^m + \left(\sum_1^{n-1} A - 2\right) (F_n+B)^m$$

$$= A_1(F_n - F_1 + B)^m + A_2(F_n - F_2 + B)^m + A_3(F_n - F_3 + B)^m \cdots A_{n-2}(F_n - F_{n-2} + B)^m + A_{n-1}(F_n - F_{n-1} + B)^m$$
$$+ \left(\sum_1^{n-1} A - 2\right) B^m \quad \text{(where } m = 1, 2\text{)}.$$

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