NUMERATOR POLYNOMIAL COEFFICIENT ARRAY FOR THE CONVOLVED FIBONACCI SEQUENCE

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1. INTRODUCTION

In [1], [2], and [3], Hoggatt and Bicknell discuss the numerator polynomial coefficient arrays associated with the row generating functions for the convolution arrays of the Catalan sequence and related sequences. In [4], Hoggatt and Bergum examine the irreducibility of the numerator polynomials associated with the row generating functions for the convolution arrays of the generalized Fibonacci sequence $\left\{H_n\right\}_{n=1}^{\infty}$ defined recursively by

(1.1) $$H_1 = 1, \quad H_2 = P, \quad H_n = H_{n-1} + H_{n-2}, \qquad n \geq 3,$$

where the characteristic $P^2 - P - 1$ is a prime. The coefficient array of the numerator polynomials is also examined. The purpose of this paper is to examine the numerator polynomials and coefficient array related to the row generating functions for the convolution array of the Fibonacci sequence. That is, we let $P = 1$.

2. THE FIBONACCI ARRAY

We first note that many of the results of this section could be obtained from [4] by letting $P = 1$.

The convolution array, written in rectangular form, for the Fibonacci sequence is

Table 1
Convolution Array for the Fibonacci Sequence

1	1	1	1	1	1	1	1	...
1	2	3	4	5	6	7	8	...
2	5	9	14	20	27	35	44	...
3	10	22	40	65	98	140	192	...
5	20	51	105	190	315	490	726	...
8	38	111	256	511	924	1554	2472	...
13	71	233	594	1295	2534	4578	7776	...
21	130	474	1324	3130	6588	12,720	22,968	...
...	...	...	...	...	...	...	...	...

The generating function $C_m(x)$ for the m^{th} column of the convolution array is given by

(2.1) $$C_m(x) = (1 - x - x^2)^{-m}$$

and it is obvious that

(2.2) $$C_m(x) = (x + x^2)C_m(x) + C_{m-1}(x).$$

Hence, if $R_{n,m}$ is the element in the n^{th} row and m^{th} column of the convolution array then the rule of formation for the convolution array is

(2.3) $$R_{n,m} = R_{n-1,m} + R_{n-2,m} + R_{n,m-1}$$

which is representable pictorially by

	w
	v
u	x

where

(2.4) $$x = u + v + w.$$

If $R_m(x)$ is the generating function for the m^{th} row of the convolution array then we see by (2.3) and induction that

(2.5) $$R_1(x) = \frac{1}{1-x}$$

(2.6) $$R_2(x) = \frac{1}{(1-x)^2}$$

and

(2.7) $$R_m(x) = \frac{N_{m-1}(x) + (1-x)N_{m-2}(x)}{(1-x)^m} = \frac{N_m(x)}{(1-x)^m}, \qquad m \geqslant 3$$

with $N_m(x)$ a polynomial of degree

$$\left[\frac{m-1}{2}\right],$$

where $[\ ]$ is the greatest integer function.

The first few numerator polynomials are found to be

$$N_1(x) = 1$$
$$N_2(x) = 1$$
$$N_3(x) = 2 - x$$
$$N_4(x) = 3 - 2x$$
$$N_5(x) = 5 - 5x + x^2$$
$$N_6(x) = 8 - 10x + 3x^2$$
$$N_7(x) = 13 - 20x + 9x^2 - x^3$$
$$N_8(x) = 21 - 38x + 22x^2 - 4x^3 .$$

Recording our results by writing the triangle of coefficients for these polynomials, we have

Table 2

Coefficients of Numerator Polynomials $N_m(x)$

1			
1			
2	−1		
3	−2		
5	−5	1	
8	−10	3	
13	−20	9	−1
21	−38	22	−4

Examining Tables 1 and 2, it appears as if there exists a relationship between the rows of Table 2 and the rising diagonals of Table 1. In fact, we shall now show that

(2.8) $$N_m(x) = \sum_{n=1}^{k} R_{m-2n+2,n}(-x)^{n-1}, \qquad m \geqslant 2,$$

where

$$k = \left[\frac{m+1}{2}\right].$$

It is obvious from (2.5), (2.6), and (2.7) that the constant coefficient of $N_m(x)$ is F_m for all $m \geqslant 1$, where F_m is the m^{th} Fibonacci number. Furthermore, the rule of formation for the elements in Table 2 is given pictorially by

w	v
	u
	x

where

(2.9) $$x = \pm(u + v - w)$$

with the sign chosen according as x is in an even or odd numbered column.

Letting $G_m(x)$ be the generating function for the m^{th} column and using (2.9) with induction, we see that

(2.10) $$G_m(x) = \left(\frac{-1}{1-x-x^2}\right) G_{m-1}(x) = \frac{(-1)^{m-1}}{(1-x-x^2)^m} = (-1)^{m-1}C_m(x).$$

Equations (2.9) and (2.10) show that the columns of Table 2 are the columns of Table 1 shifted downward by the value of $2(m-1)$ and having the sign $(-1)^{m-1}$. Hence, Eq. (2.8) is proved.

Adding along rising diagonals of Table 2 is equivalent to

(2.11) $$\sum_{k=0}^{\infty} x^{3k}G_{k+1}(x) = \left(\frac{1}{1-x-x^2}\right) \div \left(1 + \frac{x^3}{1-x-x^2}\right) = (1-x-x^2+x^3)^{-1}$$

which is the generating function for the sequence defined by

(2.12) $$S_n = \begin{cases} \left[\frac{n}{2}\right], & n \text{ even} \\ \left[\frac{n}{2}\right] + 1, & n \text{ odd} \end{cases}.$$

Letting

(2.13) $$G_k^*(x) = (1-x-x^2)^{-k},$$

we see that adding along rising diagonals with all signs positive is equivalent to

(2.14) $$\sum_{k=0}^{\infty} x^{3k}G_{k+1}^*(x) = \left(\frac{1}{1-x-x^2}\right) \div \left(1 - \frac{x^3}{1-x-x^2}\right) = \frac{1}{1-x-x^2-x^3}$$

which is the generating function for the sequence of Tribonacci numbers.

Since

(2.15) $$\sum_{k=0}^{\infty} x^{2k}G_{k+1}(x) = \left(\frac{1}{1-x-x^2}\right) \div \left(1 + \frac{x^2}{1-x-x^2}\right) = (1-x)^{-1},$$

we know that the row sums of Table 2 are always one. This fact can also be shown in the following way. From (2.7), we determine that the generating function for the polynomials $N_m(x)$ is

(2.16) $$\frac{1}{1-\lambda-(1-x)\lambda^2} = \sum_{k=0}^{\infty} N_{k+1}(x)\lambda^k.$$

Letting $x = 1$, we have

(2.17) $$\frac{1}{1-\lambda} = \sum_{k=0}^{\infty} N_{k+1}(1)\lambda^k$$

so that $N_k(1) = 1$ for all $k \geq 1$. When $x = 0$, we obtain an alternate proof that the constant coefficient of $N_m(x)$ is F_m.

Row sums with all signs positive is given by

$$\sum_{k=0}^{\infty} x^{2k} G_k^*(x) = \left(\frac{1}{1-x-x^2}\right) \div \left(1 - \frac{x^2}{1-x-x^2}\right) = (1-x-2x^2)^{-1} \tag{2.18}$$

which is the generating function for the sequence defined recursively by

$$T_1 = 1, \quad T_2 = 1, \quad T_n = T_{n-1} + 2T_{n-2}, \quad n \geq 3. \tag{2.19}$$

It is interesting to observe that by letting $x = -1$ in (2.16) we have $N_k(-1) = T_k$ for $k \geq 1$.

Adding along falling diagonals is equivalent to

$$\sum_{k=0}^{\infty} x^k G_{k+1}(x) = \frac{1}{1-x^2} \tag{2.20}$$

which is the generating function for the sequence defined by

$$S_n = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \tag{2.21}$$

In conclusion, we note that the sum of falling diagonals with all elements positive is equivalent to

$$\sum_{k=0}^{\infty} x^k G_{k+1}^*(x) = \frac{1}{1-2x-x^2} \tag{2.22}$$

which is the generating function for the sequence of Pellian numbers defined recursively by

$$P_1 = 1, \quad P_2 = 2, \quad P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3. \tag{2.23}$$

3. PROPERTIES OF $\{N_m(x)\}_{m=1}^{\infty}$

The main purpose of this section is to show that if $m \geq 5$ then $N_m(x)$ is irreducible if and only if m is a prime. The irreducibility of $N_m(x)$ for $1 \leq m \leq 5$ is obvious.

By standard finite difference techniques, it can be shown that the auxiliary polynomial associated with

$$\{N_m(x)\}_{m=1}^{\infty}$$

is

$$\lambda^2 - \lambda - (1-x) = 0 \tag{3.1}$$

whose roots are

$$\lambda_1 = \frac{1+\sqrt{5-4x}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5-4x}}{2}. \tag{3.2}$$

Using (3.1) and induction, we have

$$N_m(x) = \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2}, \quad m \geq 1. \tag{3.3}$$

Since $\lambda_1\lambda_2 = x - 1$, we can use (3.3) to show that

$$N_{m+n+1}(x) = N_{m+1}(x)N_{n+1}(x) + (1-x)N_m(x)N_n(x), \quad m \geq 1, \quad n \geq 1. \tag{3.4}$$

Following the arguments of Hoggatt and Long which can be found in [6], we obtain the following results.

$$(1-x, N_m(x)) = 1, \quad m \geq 1. \tag{3.5}$$

$$(N_m(x), N_{m+1}(x)) = 1, \quad m \geq 1. \tag{3.6}$$

(3.7) If $m \geq 3$ then $N_m(x) | N_n(x)$ if and only if $m | n$.

(3.8) Let $m \geq 5$. If $N_m(x)$ is irreducible then m is a prime.

(3.9) For $m \geq 1$, $n \geq 1$, $(N_m(x), N_n(x)) = N_{(m,n)}(x)$.

Substituting (3.2) into (3.3) and expanding by the binomial theorem, we obtain the following.

(3.10) $N_{2n+1}(x)$ is a monic polynomial of degree n.

(3.11) $4^n N_{2n+1}(x) \equiv 2n+1 \pmod{5-4x}$.

Let p be an odd prime, say $p = 2n+1$. By expanding (3.3) and collecting like powers of x, we obtain

$$(3.12)\quad N_p(x) = \frac{1}{2^{p-1}} \sum_{m=0}^{n} \sum_{j=m}^{n} \binom{p}{2j+1} \binom{j}{m} 5^{j-m}(-4x)^m \equiv \sum_{m=0}^{n} \binom{n}{m} 5^{n-m}(-4x)^m \pmod p$$
$$\equiv (5-4x)^{\frac{p-1}{2}} \pmod p .$$

In order to prove the converse of (3.8), we present the following argument.

Suppose that for some prime p, $p > 5$, $N_p(x)$ is reducible. Then, by (3.10), there exist two monic polynomials such that

$$N_p(x) = f(x)g(x)$$

or

$$N_p(x^2) = f(x^2)g(x^2).$$

Since all the powers of $f(x^2)$ and $g(x^2)$ are even, we can use the division algorithm to obtain

$$4^t f(x^2) = \ell_1(x)(5-4x^2) + h$$

and

$$4^q g(x^2) = \ell_2(x)(5-4x^2) + g \quad ,$$

where t and q are respectively the degrees of $f(x)$ and $g(x)$ and h and g are integers.

By (3.11), we see that

$$(3.13)\quad 4^{\frac{p-1}{2}} f(x^2)g(x^2) \equiv p \equiv hg \pmod{5-4x^2}.$$

Hence, we assume without loss of generality that $h = \pm p$ and $g = \pm 1$.

If $p \equiv \pm 2 \pmod 5$ then 5 is a quadratic nonresidue so that $5-4x^2$ is irreducible in the unique factorization domain $Z_p[x]$. Hence, by (3.12), we conclude that $g(x) \equiv (5-4x^2)^k \pmod p$ for some integer k. If $p \equiv \pm 1 \pmod 5$ then 5 is a quadratic residue so that

$$(5-4x^2) = (a-2x)(a+2x) \text{ in } Z_p[x] \text{ with } a^2 \equiv 5 \pmod p .$$

Therefore, by (3.12),

$$g(x^2) \equiv (a-2x)^{k_1}(a+2x)^{k_2} \pmod p$$

for some integers k_1 and k_2. However, $g(x^2)$ is even so that $k_1 = k_2$. In both cases, there exists an integer k such that

$$(3.14)\quad g(x^2) = \ell_3(x)p + (5-4x^2)^k .$$

Since $\ell_3(x)$ is obviously even, we know that

$$(3.15)\quad 4^q \ell_3(x) \equiv c \pmod{5-4x^2}$$

for some integer c so that

$$(3.16)\quad 4^q g(x^2) \equiv \pm 1 \equiv pc \pmod{5-4x^2}$$

which is impossible. Hence, $N_p(x)$ is irreducible.

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LETTER TO THE EDITOR

October 13, 1975

Dear Professor Hoggatt:

It was with some surprise that I read Miss Ada Booth's article "Idiot's Roulette Revisited" in the April 1975 issue of *The Fibonacci Quarterly.* The problem she discusses—given N places circularly arranged and successively casting out the C^{th} place, determine which will be the last remaining place—is quite old and commonly referred to as the Josephus problem. The name alludes to a passage in the writings of Flavius Josephus [7], a Jewish historian who relates how after the fall of Jotapata, he and forty other Jews took refuge in a nearby cave, only to be discovered by the Romans. In order to avoid capture, everyone in the group, save Josephus, resolved on mass suicide. At Josephus' suggestion, lots were drawn, and as each man's lot came up, he was killed. By means not made clear in the passage, Josephus ensured that the lots of himself and one other were the last to come up, at which point he persuaded the other man that they should surrender to Vespasian.

Bachet [2], in one of the earliest works on recreational mathematics, proposed a definite mechanism by which this could have been accomplished: all forty-one people are placed in a circle, Josephus placing himself and the other man at the 16^{th} and 31^{st} places; every third person is then counted off and killed. This is, of course, a special case of the question Miss Booth considers.

Miss Booth's iterative solution to the general problem was apparently first discovered by Euler [5] in 1771 and then rediscovered by P. G. Tait [9], the English physicist and mathematician, in 1898. Tait points out that the method enables one to calculate the last r places to be left, not merely the last as in Miss Booth's article. Although Euler and Tait content themselves with demonstrating how the iterative solution works and do not actually derive the formula for Miss Booth's sequence of "subtraction numbers," in the 1890's Schubert and Busche [8, 4] derived a formula for this sequence (slightly modified) via a wholly different attack on the problem ("Oberreihen"). (Ahrens [1] has an excellent description of this work, as well as a comprehensive review of the history of the problem. Ball and Coxeter [3] briefly touch on the problem but omit any mention of the work of Schubert and Busche.)

[Continued on Page 51.]