THE SAALSCHÜTZIAN THEOREMS

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1. Saalschütz's theorem reads

$$\sum_{k=0}^{n} \frac{(-n)_k(a)_k(b)_k}{k!(c)_k(d)_k} = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \tag{1.1}$$

where

$$c+d = -n+a+b+1 \tag{1.2}$$

and

$$(a)_k = a(a+1)\cdots(a+k-1), \qquad (a)_0 = 1.$$

The theorem has many applications. For example, making use of (1.1), one can prove [3, §6], [7, p. 41]

$$\sum_{k=0}^{n} \binom{n}{k}^3 x^k = \sum_{2j\le n} \frac{(n+j)!}{(j!)^3(n-j)!} x^j(1+x)^{n-2j}. \tag{1.3}$$

In particular, for $x = 1$, (1.3) reduces to

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{2j\le n} \frac{(n+1)!}{(j!)^3(n-j)!} 2^{n-2j}, \tag{1.4}$$

a result due to MacMahon. For $x = -1$, (1.3) yields Dixon's theorem:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{(n!)^3}. \tag{1.5}$$

Saalschütz's theorem is usually proved (see for example [2, p. 9], [6, p. 86], [8, p. 48]) by showing that it is a corollary of Euler's theorem for the hypergeometric function:

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x), \tag{1.6}$$

where as usual

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} x^n.$$

As for (1.6), the usual method of proof is by making use of the hypergeometric differential equation.

The writer [3, §6] has given an inductive proof of (1.1). We shall now show how to prove the theorem by using only Vandermonde's theorem

$$F(-n,a;c;1) = \frac{(c-a)_n}{(c)_n}. \tag{1.7}$$

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We then show how the q-analog of (1.1) can be proved in an analogous manner (for statement of the q-analog see §5 below). Finally, in §6, we prove a q-analog of (1.5).

2. To begin with, we note that (1.4) is implied by the familiar formula

$$\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}, \tag{2.1}$$

where a,b are non-negative integers. Since each side of (2.1) is a polynomial in $a,b,$ it follows that (2.1) holds for arbitrary $a,b.$ Replacing a by $-a$ and b by $c+n-1,$ (2.1) becomes

$$\sum_{k=0}^{n} \binom{-a}{k}\binom{c+n-1}{n-k} = \binom{c-a+n-1}{n},$$

that is,

$$\sum_{k=0}^{n} (-1)^k \frac{(a)_k(c+k)_{n-k}}{k!(n-k)!} = \frac{(c-a)_n}{n!}.$$

This is the same as

$$\sum_{k=0}^{n} \frac{(-n)_k(a)_k}{k!(c)_k} = \frac{(c-a)_n}{(c)_n}, \tag{2.2}$$

so that we have proved (1.4).

Now, by (2.2),

$$\sum_{n=0}^{\infty} \frac{(c-a)_n(b)_n}{n!(c)_n} x^n = \sum_{n=0}^{\infty} \frac{(b)_n}{n!} x^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_k}{(c)_k} = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(b)_k}{k!(c)_k} x^k \sum_{n=0}^{\infty} \frac{(b+k)_n}{n!} x^n$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(b)_k}{k!\,(c)_k} x^k(1-x)^{-b-k},$$

so that

$$\sum_{n=0}^{\infty} \frac{(c-a)_n(b)_n}{n!(c)_n} x^n = (1-x)^{-b} \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} \left(\frac{x}{x-1}\right)^k. \tag{2.3}$$

We have accordingly proved the well-known formula

$$F\left(a,b;c;\frac{x}{x-1}\right) = (1-x)^b F(c-a,b;c;x). \tag{2.4}$$

In the next place, by (2.3),

$$\sum_{n=0}^{\infty} \frac{(c-a)_n(c-b)_n}{n!(c)_n} x^n = (1-x)^{b-c} \sum_{k=0}^{\infty} \frac{(a)_k(c-b)_k}{k!(c)_k} \left(\frac{x}{x-1}\right)^k = (1-x)^{b-c} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \left(\frac{x}{x-1}\right)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(b)_j}{(c)_j}$$

$$= (1-x)^{b-c} \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{j!(c)_j} \left(\frac{x}{1-x}\right)^j \sum_{k=0}^{\infty} \frac{(a+j)_k}{k!} \left(\frac{-x}{1-x}\right)^k =$$

$$= (1-x)^{b-c} \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{j!(c)_j} \left(\frac{x}{1-x}\right)^j \left(1+\frac{x}{1-x}\right)^{-a-j} = (1-x)^{a+b-c} \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{j!\,(c)_j} x^j .$$

This evidently proves (1.6).

To see that (1.1) and (1.6) are equivalent, consider

$$(1-x)^{a+b-c} F(a,b;c;x) = \sum_{j=0}^{\infty} \frac{(c-a-b)_j}{j!} x^j \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} x^k$$

$$= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{(a)_k (b)_k (c-a-b)_{n-k}}{k!(c)_k (n-k)!} .$$

Since

$$(a)_{n-k} = \frac{(a)_n}{(a+n-k)\cdots(a+n-1)} = (-1)^k \frac{(a)_n}{(-a-n+1)_n} ,$$

it follows that

$$(1-x)^{a+b-c} F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(c-a-b)_n}{n!} x^n \sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!(c)_k (a+b-c-n+1)_k}$$

Hence (1.6) is equivalent to

$$\frac{(c-a-b)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (a)_k (b)_k}{k!(c)_k (a+b-c-n+1)_k} = \frac{(c-a)_n (c-b)_n}{n!(c)_n} ,$$

which is itself equivalent to (1.1).

3. It may be of interest to remark that (2.4) is a special case of the following identity:

$$\sum_{r=0}^{\infty} \frac{(a)_r}{r!} \lambda_r x^r = (1-x)^{-a} \sum_{r=0}^{\infty} \frac{(a)_r}{r!} \mu_r \left(\frac{x}{x-1}\right)^r , \tag{3.1}$$

where

$$\mu_r = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \lambda_s , \qquad \lambda_r = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \mu_s . \tag{3.2}$$

Indeed

$$(1-x)^{-a} \sum_{r=0}^{\infty} \frac{(a)_r}{r!} \mu_r \left(\frac{x}{x-1}\right)^r = \sum_{r=0}^{\infty} (-1)^r \frac{(a)_r}{r!} \mu_r x^r (1-x)^{-a-r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(a)_r}{r!} \mu_r x^r \sum_{s=0}^{\infty} \frac{(a+r)_s}{s!} x^s$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} \mu_r$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \lambda_n x^n .$$

For $\lambda_r = (b)_r/(c)_r$, (3.1) reduces to an identity equivalent to (2.4). For

$$\lambda_r = \frac{1}{c+r}, \qquad \mu_r = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{1}{c+s} = \frac{r!}{(c)_{r+1}}$$

and (3.1) becomes

$$\sum_{r=0}^{\infty} \frac{(a)_r}{r!} \frac{x^r}{c+r} = (1-x)^{-a} \sum_{r=0}^{\infty} \frac{(a)_r}{(c)_{r+1}} \left(\frac{x}{x-1}\right)^r . \tag{3.3}$$

4. We turn next to the q-analog of Saalshütz's theorem. We shall use the following notation. Put

$$(a)_n = (a)_{n,q} = (1-a)(1-qa)\cdots(1-q^{n-1}a), \qquad (a)_0 = 1; \tag{4.1}$$

in particular

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n), \qquad (q)_0 = 1.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}} ;$$

it occurs in the q-binomial theorem

$$(x)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} x^k .$$

We also put

$$e(x) = e(x,q) = \prod_{n=0}^{\infty} (1-q^n x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{(q)_n} ,$$

where $|q| < 1$, $|x| < 1$. A more general result used below is

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \frac{e(x)}{e(ax)} . \tag{4.2}$$

We shall also use the identity

$$\frac{1}{e(x)} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} \frac{x^n}{(q)_n} .$$

For completeness we sketch the proof of (4.2). Put

$$f(x) = \frac{e(x)}{e(ax)} = \sum_{n=0}^{\infty} A_n x^n .$$

Since $e(qx) = (1-x)e(x)$, it follows that

$$f(qx) = \frac{1-x}{1-ax} f(x),$$

so that

$$(1-x) \sum_{n=0}^{\infty} A_n x^n = (1-ax) \sum_{n=0}^{\infty} A_n q^n x^n .$$

This gives

$$A_n - A_{n-1} = q^n A_n - q^{n-1} a A_{n-1}, \qquad (1-q^n)A_n = (1-q^{n-1}a)A_{n-1}.$$

Since $A_0 = 1$, we get

$$A_n = \frac{1-q^{n-1}a}{1-q^n} A_{n-1} = \frac{(a)_n}{(q)_n},$$

thus proving (4.2).

We shall also require the following formulas:

$$\sum_{k=0}^{n} (-1)^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k}{(c)_k} q^{\frac{1}{2}k(k-1)} (c/a)^k = \frac{(c/a)_n}{(c)_n}, \tag{4.3}$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k}{(c)_k} q^{\frac{1}{2}k(k+1)-nk} = \frac{(c/a)_n}{(c)_n} a^n, \tag{4.4}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(a)_n q^{\frac{1}{2}n(n-1)}}{(q)_n (c)_n} (c/a)^n = \frac{e(c)}{e(c/a)}. \tag{4.5}$$

To prove (4.3), we note first that it follows from (4.2) and the evident identity

$$\frac{e(x)}{e(ax)} \frac{e(ax)}{e(abx)} = \frac{e(x)}{e(abx)}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (b)_{n-k} a^{n-k} = (ab)_n. \tag{4.6}$$

Replacing by $= q^{-n+1}/c$, this becomes

$$\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (q^{-n+1}/c)_{n-k} a^{n-k} = (q^{-n+1}a/c)_n. \tag{4.7}$$

Now

$$\begin{aligned} (q^{-n+1}a/c)_n &= \left(1 - \frac{q^{-n+1}a}{c}\right)\left(1 - \frac{q^{-n+2}a}{c}\right) \cdots \left(1 - \frac{a}{c}\right) \\ &= (-1)^n q^{-\frac{1}{2}n(n-1)} (a/c)^n (c/a)_n \,; \end{aligned}$$

similarly

$$\begin{aligned} (q^{-n+1}/c)_{n-k} &= \left(1 - \frac{q^{-n+1}}{c}\right)\left(1 - \frac{q^{-n+2}}{c}\right) \cdots \left(1 - \frac{q^{-k}}{c}\right) \\ &= (-1)^{n-k} q^{-\frac{1}{2}n(n-1)+\frac{1}{2}k(k-1)} c^{-n+k} (q^k c)_{n-k} \\ &= (-1)^{n-k} q^{-\frac{1}{2}n(n-1)+\frac{1}{2}k(k-1)} c^{-n+k} (c)_n/(c)_k. \end{aligned}$$

Hence (4.7) becomes

$$\sum_{k=0}^{n} (-1)^k q^{\frac{1}{2}k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k}{(c)_k} (c/a)^k = \frac{(c/a)_n}{(c)_n},$$

so that we have proved (3.3).

To prove (4.4), rewrite (4.6) in the form

(4.8)
$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (b)_{n-k} b^k = (ab)_n .$$

Then exactly as above

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (q^{-n+1}/c)_{n-k} (q^{-n+1}/c)^k = (q^{-n+1}a/c)_n ,$$

which reduces to

$$\sum_{k=0}^{n} (-1)^k q^{\frac{1}{2}k(k+1)-nk} (a)_k (q^k c)_{n-k} = (c/a)_n a^n .$$

As for (4.5), we take

$$\frac{e(a)}{e(c)} \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n q^{\frac{1}{2}n(n-1)}}{(q)_n (c)_n} (c/a)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}(c/a)^n}{(q)_n} \frac{e(q^n a)}{e(q^n c)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}(c/a)^n}{(q)_n} \sum_{k=0}^{\infty} \frac{(c/a)_k}{(q)_k} (q^n a)^k$$

$$= \sum_{k=0}^{\infty} \frac{(c/a)_k}{(q)_k} a^k \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}(q^k c/a)^n}{(q)_n}$$

$$= \sum_{k=0}^{\infty} \frac{(c/a)_k}{(q)_k} a^k \frac{1}{e(q^k c/a)}$$

$$= \frac{1}{e(c/a)} \sum_{k=0}^{\infty} \frac{a^k}{(q)_k} = \frac{e(a)}{e(c/a)} .$$

This evidently proves (4.5).

5. The q-analog of Saalschütz's theorem reads

(5.1)
$$\sum_{k=0}^{n} \frac{(q^{-n})_k (a)_k (b)_k}{(q)_k (c)_k (d)_k} q^k = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n} ,$$

where

(5.2)
$$cd = q^{-n+1} ab .$$

The theorem is usually proved (see for example [2, p. 68], [8, p. 96]) as a special case of a much more elaborate result for generalized basic hypergeometric series. We shall give a proof analogous to the proof in §2 of the ordinary Saalschütz theorem.

Making use of (5.2), we may rewrite (5.1) as follows.

(5.3)
$$\sum_{k=0}^{n} \frac{(q^{-n})_k (a)_k (b)_k}{(q)_k (c)_k (q^{-n+1}ab/c)_k} q^k = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n} .$$

Since

$$(q^{-n})_k = (-1)^k q^{\frac{1}{2}k(k-1)-nk} (q)_n / (q)_{n-k} ,$$

$$(q^{-n+1}ab/c)_k = (-1)^k q^{\frac{1}{2}k(k+1)-nk} (ab/c)^k (c/ab)_n (c/ab)_{n-k} ,$$

(5.3) becomes

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k (b)_k}{(c)_k (c/ab)_n} (c/ab)_{n-k} \left(\frac{c}{ab}\right)^k = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n} .$$

It follows that

$$\sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} x^n = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(q)_k (c)_k} \left(\frac{cx}{ab}\right)^k \sum_{n=0}^{\infty} \frac{(c/ab)_n}{(q)_n} x^n .$$

Hence, by (4.2), we have

(5.4)
$$\sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} x^n = \frac{e(x)}{e(cx/ab)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(q)_k (c)_k} \left(\frac{cx}{ab}\right)^k ,$$

an identity due to Heine. Clearly (5.3) and (5.4) are equivalent, so it will suffice to prove (5.4).

By (4.3),

$$\sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} x^n = \sum_{n=0}^{\infty} \frac{(c/b)_n}{(q)_n} x^n \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a)_k}{(c)_k} q^{\frac{1}{2}k(k-1)} (c/a)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k} q^{\frac{1}{2}k(k-1)} (cx/a)^k \sum_{n=0}^{\infty} \frac{(q^k c/b)_n}{(q)_n} x^n$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k} q^{\frac{1}{2}k(k-1)} (cx/a)^k \frac{e(x)}{e(q^k cx/b)}$$

(5.5)
$$= \frac{e(x)}{e(cx/b)} \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k (cx/b)_k} q^{\frac{1}{2}k(k-1)} (cx/a)^k .$$

Next, using (4.4) we get

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k (cx/b)_k} q^{\frac{1}{2}k(k-1)} (cx/a)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k q^{\frac{1}{2}k(k-1)}}{(q)_k (cx/b)_k} (cx/ab)^k \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} \frac{(b)_j}{(c)_j} q^{\frac{1}{2}j(j+1)-jk}$$

$$= \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(q)_j (c)_j (cx/b)_j} (cx/ab)^j \sum_{k=0}^{\infty} (-1)^k \frac{(q^j a)_k}{(q)_k (q^j cx/b)_k} q^{\frac{1}{2}k(k-1)} (cx/ab)^k .$$

By (4.5) the inner sum is equal to

$$\frac{e(q^j cx/b)}{e(cx/ab)} = \frac{e(cx/b)}{e(cx/ab)} (cx/b)_j .$$

Hence we have

$$\sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c/b)_k}{(q)_k (c)_k (cx/b)_k} q^{\frac{1}{2}k(k-1)} (cx/ab)^k \tag{5.6}$$

$$= \frac{e(cx/b)}{e(cx/ab)} \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(q)_j (c)_j} (cx/ab)^j .$$

Combining (5.5) and (5.6), we get

$$\sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n}{(q)_n (c)_n} x^n = \frac{e(x)}{e(cx/ab)} \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(q)_j (c)_j} (cx/ab)^j . \tag{5.7}$$

Thus we have proved (5.4) and so have proved (5.1).

6. We now give an application of (5.1). Making some changes in notation, (5.1) can be written in the following form.

$$\sum_{j=0}^{k} \frac{(q^{-k})_j (q^k a)_j (qbc/a)_j}{(q)_j (qb)_j (qc)_j} q^j = \frac{(a/b)_k (a/c)_k}{(qb)_k (qc)_k} \left(\frac{qbc}{a} \right)^k . \tag{6.1}$$

It follows that

$$\sum_{k=0}^{\infty} \frac{(a)_k (a/b)_k (a/c)_k}{(q)_k (qb)_k (qc)_k} \left(\frac{qbcx}{a} \right)^k = \sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} x^k \sum_{j=0}^{k} \frac{(q^{-k})_j (q^k a)_j (qbc/a)_j}{(q)_j (qb)_j (qc)_j} q^j$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{(q)_k} \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a)_{j+k} (qbc/a)_j}{(qb)_j (qc)_j} q^{\frac{1}{2}j(j+1)-jk}$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{(a)_{2j} (qbc/a)_j}{(q)_j (qb)_j (qc)_j} q^{-\frac{1}{2}j(j-1)} x^j \sum_{x=0}^{\infty} \frac{(q^{2j} a)_k}{(q)_k} (q^{-j} x)^k .$$

We now take $a = q^{-2m}$ and replace x by $q^m x$, where m is a non-negative integer. The above identity becomes

$$\sum_{k=0}^{2m} \frac{(q^{-2m})_k (q^{-2m}/b)_k (q^{-2m}/c)_k}{(q)_k (qb)_k (qc)_k} (q^{3m+1} bcx)^k \tag{6.2}$$

$$= \sum_{j=0}^{m} (-1)^j \frac{(q^{-2m})_{2j} (q^{2m+1} bc)_j}{(q)_j (qb)_j (qc)_j} q^{mj - \frac{1}{2}j(j-1)} x^j \sum_{k=0}^{2m-2j} \frac{(q^{-2m+2j})_k}{(q)_k} (q^{m-j} x)^k .$$

The inner sum on the right is equal to

$$\sum_{k=0}^{2m-2j} (-1)^k \begin{bmatrix} 2m-2j \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} (q^{-m+j} x)^k = (q^{-m+j} x)_{2m-2j} .$$

We have therefore proved the identity

(6.3)
$$\sum_{k=0}^{2m} \frac{(q^{-2m})_k (q^{-2m}/b)_k (q^{-2m}/c)_k}{(q)_k (qb)_k (qc)_k} (q^{3m+1}bcx)^k$$

$$= \sum_{j=0}^{m} (-1)^j \frac{(q^{-2m})_{2j} (q^{-2m+1}/bc)_j}{(q)_j (qb)_j (qc)_j} q^{mj-\frac{1}{2}j(j-1)} x^j (q^{-m+j}x)_{2m-2j} .$$

For $x = 1$, (6.3) becomes

(6.4)
$$\sum_{k=0}^{2m} \frac{(q^{-2m})_k (q^{-2m}/b)_k (q^{-2m}/c)_k}{(q)_k (qb)_k (qc)_k} (q^{3m+1}bc)^k$$

$$= (-1)^m \frac{(q)_{2m} (q^{-2m+1}/bc)_m}{(q)_m (qb)_m (qc)_m} q^{-\frac{1}{2}m(3m+1)} .$$

In particular, for $b = c = 1$, (6.4) reduces to

(6.5)
$$\sum_{k=0}^{2m} (-1)^k \begin{bmatrix} 2m \\ k \end{bmatrix}^3 q^{\frac{3}{2}(m-k)^2 + \frac{1}{2}(m-k)} = (-1)^m \frac{(q)_{3m}}{((q)_m)^3} ,$$

a result due to Jackson [5] and Bailey [1]. Jackson's more general results can also be proved [4] using the q-analog of Saalschütz's theorem.

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