A SUMMATION IDENTITY

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The purpose of this note is to generalize the following two well known formulas:

(1) $$\sum_{i=k}^{n}\sum_{j=k}^{n-i+k} = \sum_{j=k}^{n}\sum_{i=k}^{n-j+k}$$

(2) $$\sum_{i=k}^{n}\sum_{j=k}^{i} = \sum_{j=k}^{n}\sum_{i=j}^{n} \quad .$$

These are double summation operators, and the equality means that when either operator acts on an arbitrary doubly subscripted sequence, the same result is obtained.

To show how these formulas can be compounded, and to motivate the general result to follow, we offer the following example. Each equality is justified by (1), (2), or the fact that the operators may commute when the subscripts involved are independent of each other. Note that the use of the parenthesis is to indicate which pair of operators is being permuted, and is not to imply any sort of associative law.

$$\begin{aligned}\left(\sum_{i_1=k}^{n}\sum_{i_2=k}^{n-i_1+k}\right)\sum_{i_3=i_2}^{n}\sum_{i_4=k}^{i_3} &= \sum_{i_2=k}^{n}\left(\sum_{i_1=k}^{n-i_2+k}\sum_{i_3=i_2}^{n}\right)\sum_{i_4=k}^{i_3} = \sum_{i_2=k}^{n}\sum_{i_3=i_2}^{n}\left(\sum_{i_1=k}^{n-i_2+k}\sum_{i_4=k}^{i_3}\right)\\ &= \left(\sum_{i_2=k}^{n}\sum_{i_3=i_2}^{n}\right)\sum_{i_4=k}^{i_3}\sum_{i_1=k}^{n-i_2+k} = \sum_{i_3=k}^{n}\left(\sum_{i_2=k}^{i_3}\sum_{i_4=k}^{i_3}\right)\sum_{i_1=k}^{n-i_2+k}\\ &= \left(\sum_{i_3=k}^{n}\sum_{i_4=k}^{i_3}\right)\sum_{i_2=k}^{i_3}\sum_{i_1=k}^{n-i_2+k} = \sum_{i_4=k}^{n}\sum_{i_3=i_4}^{n}\sum_{i_2=k}^{i_3}\sum_{i_1=k}^{n-i_2+k} \quad .\end{aligned}$$

If we examine the resulting equality of the first and last operators, we see that order is reversed with respect to the i_j's, i_4 replaces i_1 in the first factor, and the other factors are exchanged according to the schemes:

(3) $$\sum_{i=k}^{n-j+k} \longleftrightarrow \sum_{j=k}^{n-i+k}$$

(4) $$\sum_{i=j}^{n} \longleftrightarrow \sum_{j=k}^{i}$$

Theorem. Consider an operator of the form

$$\sum_{i_1=k}^{n}\prod_{j=2}^{t}\sum_{i_j=h_j}^{n_j} ,$$

where the pairs h_j, n_j are of the following types:

(5) $$h_j = k \quad \text{and} \quad n_j = n - i_{j-1} + k$$

(6) $$h_j = k \quad \text{and} \quad n_j = i_{j-1}$$

(7) $$h_j = i_{j-1} \quad \text{and} \quad n_j = n \ .$$

Then

$$\sum_{i_1=k}^{n} \prod_{j=2}^{t} \sum_{i_j=h_j}^{n_j} = \sum_{i_t=k}^{n} \prod_{j=1}^{t-1} \sum_{i_{t-j}},$$

where each factor on the right (after the first) has been exchanged according to the scheme (3), (4).

Proof. Inductively, suppose that the theorem is true for $(t-1)$ factors. We have

(8) $$\sum_{i_1=k}^{n} \prod_{j=2}^{t} \sum_{i_j=h_j}^{n_j} = \left(\sum_{i_1=k}^{n} \sum_{i_2=h_2}^{n_2} \right) \prod_{j=3}^{t} \sum_{i_j=h_j}^{n_j} = \sum_{i_2=k}^{n} \sum_{i_1} \prod_{j=3}^{t} \sum_{i_j=h_j}^{n_j} \quad ,$$

where the $\sum\limits_{i_1}$ factor has been transformed by (3) or (4). Now the factor $\sum\limits_{i_1}$ can commute with each $\sum\limits_{i_j=h_j}^{n_j}$, $t \geq j \geq 3$, since each of these h_j's and n_j's are independent of i_1. Hence

(9) $$\sum_{i_1=k}^{n} \prod_{j=2}^{t} \sum_{i_j=h_j}^{n_j} = \left(\sum_{i_2=k}^{n} \prod_{j=3}^{t} \sum_{i_j=h_j}^{n_j} \right) \sum_{i_1},$$

and the result follows from the induction hypothesis on the first $(t-1)$ factors.

Example.

$$\sum_{i_1=k}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_t=i_{t-1}}^{n} \binom{i_1}{k} = \sum_{i_t=k}^{n} \sum_{i_{t-1}=k}^{i_t} \cdots \sum_{i_1=k}^{i_2} \binom{i_1}{k} = \sum_{i_t=k}^{n} \sum_{i_{t-1}=k}^{i_t} \cdots \sum_{i_2=k}^{i_3} \binom{i_2+1}{k+1}$$

$$= \cdots = \sum_{i_t=k}^{n} \binom{i_t+t-1}{k+t-1} = \binom{n+t}{k+t} .$$

On the other hand, since

$$\sum_{i_2=i_1}^{n} \cdots \sum_{i_t=i_{t-1}}^{n} 1 = \binom{n-i_1+t-1}{t-1} ,$$

$$\sum_{i_1=k}^{n} \cdots \sum_{i_t=i_{t-1}}^{n} \binom{i_1}{k} = \sum_{i_1=k}^{n} \binom{i_1}{k} \sum_{i_2=i_1}^{n} \cdots \sum_{i_t=i_{t-1}}^{n} 1 = \sum_{i_1=k}^{n} \binom{i_1}{k} \binom{n-i_1+t-1}{t-1} ,$$

we have

$$\sum_{i_1=k}^{n} \binom{i_1}{k} \binom{n-i_1+t-1}{t-1} = \binom{n+t}{k+t} .$$

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