A COMBINATORIAL IDENTITY

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Let $q < p < k$ and v be positive integers, n be a nonnegative integer, $\ell_0 = 1$ and $\{\ell_1, \ell_2 \cdots\}$ be a sequence of marks. Further let $T_{k,j}$ be the Stirling numbers of the first kind defined as the coefficients of

(1) $$f(x) = \sum_{j=1}^{k} T_{k,j}x^j = x(x-1)(x-2)\cdots(x-k+1)$$

and let

(2) $$L(v,p,q) = \sum r_1 r_2 \cdots r_v \ell_{d_1}\ell_{d_2}\cdots\ell_{d_v},$$

where the summation is over all the sequences of integers $r_1, r_2, \cdots, r_v$ satisfying

$$p = r_0 \geqslant r_1 \geqslant r_2 \geqslant \cdots \geqslant r_v = p-q, \quad \text{and} \quad d_i = r_{i-1} - r_i.$$

In connection with integration of differential equations of a group, A Ran proved in his thesis [1], using analytical methods, that

(3) $$\sum_{j=1}^{k} T_{k,j} L(j+n,p,q) \equiv 0$$

identically, i.e., that on the left side of (3) the coefficient of every product $\Pi \ell_i^{\alpha_i}$ equals zero.

Here the proof of (3) is given by combinatorial methods. To begin we write (2) in the form

(4) $$L(v,p,q) = \sum{}^{*} R(v,p,q,a,\pi\ell_i^{\alpha_i}) \prod_{i\geqslant 1} \ell_i^{\alpha_i},$$

where the summation Σ^* is over all sequences of nonnegative integers $\alpha_1, \alpha_2, \cdots, \alpha_q$ satisfying $\Sigma i\alpha_i = q$, and

(5) $$a = \sum \alpha_i,$$

and prove the following

Lemma.

(6) $$R(v,p,q,a,\pi\ell_i^{\alpha_i}) = \sum_{h=0}^{q} c_h (p-h)^v,$$

where the coefficients c_h do not depend on v (but may depend on p, q, a and $\pi\ell_i^{\alpha_i}$) and are such that

(7) $$\sum_{h=0}^{q} c_h (p-h)^t = 0, \qquad t = 0, 1, \cdots, a-1.$$

Proof. The proof is given by induction on a. For $a = 1$ we have

$$R(v,p,q,1,\ell_q) = (p-q)\sum_{i=0}^{v-1} p^i (p-q)^{v-i-1} = \frac{p-q}{q}(p^v - (p-q)^v),$$

which satisfies both (6) and (7).

Suppose now that (6) and (7) are satisfied for $a = b - 1$. It is easily seen that

$$R(v,p,q,b,\pi \ell_i^{\alpha_i}) = \sum_{\eta} (p-\eta) \sum_{\beta=0}^{v-b} p^{\beta} R(v-\beta-1, p-\eta, q-\eta, b-1, \pi \ell_i^{\alpha_i}/\ell_\eta),$$

where η obtains the values of i for which $\alpha_i \geq 1$. We make use of (6) with $a = b - 1$ and in order to stress that the coefficients c_h depend on η we write them in the form $c_{\eta,h}$. We have

$$R(v,p,q,b,\pi \ell_i^{\alpha_i}) = \sum_{\eta} (p-\eta) \sum_{\beta=o}^{v-b} p^{\beta} \sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{v-\beta-1}$$

$$= \sum_{\eta} (p-\eta) \left[\sum_{h=\eta}^{q} \frac{c_{\eta,h}}{h} (p^v - (p-h)^v) - \sum_{\beta=v-b+1}^{v-1} p^{\beta} \sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{v-\beta-1} \right].$$

By (7) follows that

$$\sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{v-\beta-1} = 0$$

for every η and for $0 \leq v - \beta - 1 \leq b - 2$, i.e., for $v - b + 1 \leq \beta \leq v - 1$ and consequently

(8) $$R(v,p,q,b,\pi \ell_i^{\alpha_i}) = \sum_{\eta} (p-\eta) \cdot \sum_{h=\eta}^{q} \frac{c_{\eta,h}}{h} (p^v - (p-h)^v)$$

which proves (6) for $a = b$.

To prove (7) let us denote for every η

(9) $$D_\eta(t) = \sum_{h=\eta}^{q} \frac{c_{\eta,h}}{h} (p^t - (p-h)^t).$$

Evidently $D_\eta(0) = 0$. For $t \geq 1$ we have

$$D_\eta(t) = \sum_{h=\eta}^{q} \frac{c_{\eta,h}}{h} \cdot h \sum_{i=0}^{t-1} p^i (p-h)^{t-i-1} = \sum_{i=0}^{t-1} p^i \sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{t-i-1}.$$

By (7) with $a = b - 1$,

$$\sum_{h=\eta}^{q} c_{\eta,h}(p-h)^{t-i-1} = 0$$

for $t = 1, 2, \cdots, b-1$ and $0 \leq i \leq t-1$ and consequently $D_\eta(t) = 0$ for $0 \leq t \leq b-1$. By (6), (8) and (9),

$$\sum_{h=o}^{q} c_h(p-h)^t = R(t,p,q,b,\pi \ell_i^{\alpha_i}) = \sum_{\eta} (p-\eta) D_\eta(t) = 0, \qquad t = 0, 1, \cdots, b-1$$

which proves (7) with $a = b$.

Theorem.

$$\sum_{j=1}^{k} T_{k,j} L(j+n,p,q) \equiv 0.$$

Proof. By (4), (6) and (1) we have

$$\sum_{j=1}^{k} T_{k,j}L(j+n,p,q) = \sum_{j=1}^{k} T_{k,j} \sum_{-} \prod_{i=1}^{q} \varrho_i^{\alpha_i} \sum_{h=o}^{q} c_h(p-h)^{j+n}$$

$$= \sum^{*} \prod_{i=1}^{q} \varrho_i^{\alpha_i} \sum_{h=0}^{q} c_h(p-h)^n \sum_{j=1}^{k} T_{k,j}(p-h)^j = \sum^{*} \prod_{i=1}^{q} \varrho_i^{\alpha_i} \sum_{h=o}^{q} c_h(p-h)^n f(p-h).$$

By definition $p-h$ is an integer satisfying $1 \leqslant p-h \leqslant p \leqslant k-1$ and consequently by (1), $f(p-h)=0$ which proves the theorem.

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Much more recently (1973), Jacobczyk [6] has given new iterative procedures for determining answers to both:

(a) for each k, $1 \leqslant k \leqslant N$, which will be the k^{th} place to be cast out?

(b) for each k, $1 \leqslant k \leqslant N$, when will the k^{th} place be cast out?

(The "Oberreihen" methods described by Ahrens also provide answers to both questions.)

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