SOME OPERATIONAL FORMULAS

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1. INTRODUCTION

In this paper we consider some simple variations of the derivative and the difference operator; deriving formulas for powers and factorials.

Let $s(n,k)$ denote the Stirling number of the first kind and $S(n,k)$ denote the Stirling number of the second kind. They are defined by:

(1.1) $$(x)_n = \sum_{k=1}^{n} s(n,k)x^k$$

(1.2) $$x^n = \sum_{k=1}^{n} S(n,k)(x)_k ,$$

where

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

Substituting (1.1) in (1.2) or (1.2) in (1.1) shows that

$$a_n = \Sigma s(n,k)b_k \qquad \text{and} \qquad b_n = \Sigma S(n,k)a_k$$

are equivalent (inverse) relations.

Define

(1.3) $$A_n(x) = \sum_{k=1}^{n} s(n,k)x^k$$

(1.4) $$A^{(n)}(x) = \sum_{k=1}^{n} (-1)^{n-k}s(n,k)x^k$$

(1.5) $$B_n(x) = \sum_{k=1}^{n} S(n,k)x^k$$

(1.6) $$B^{(n)}(x) = \sum_{k=1}^{n} (-1)^{n-k}S(n,k)x^k .$$

Then $A_n(x) = (x)_n$, the falling factorial; $A^{(n)}(x) = x^{(n)}$, the rising factorial and $B_n(x)$ is the single variable Bell polynomial [3, p. 35]. We have $A_n(B(x)) = x^n = B_n(A(x))$, etc., where $(B(x))^k \equiv B_k(x)$, $(A(x))^k \equiv A_k(x)$.

We will employ the following special notation:

(1.7) $$[\theta\phi]^n = \theta^n\phi^n$$

and if

$$f_n(x) = \sum_{i=0}^{n} a_i x^i$$

then

$$f_n[\theta\phi] = \sum_{i=0}^{n} a_i[\theta\phi]^i = \sum_{i=0}^{n} a_i \theta^i \phi^i .$$

REMARK. When θ and ϕ commute or $n = 1$ then

$$[\theta\phi]^n = (\theta\phi)^n \qquad \text{and} \qquad f_n(\partial\phi) = f_n[\theta\phi] .$$

2. THE OPERATORS xD, Dx, $x\Delta$, Δx

Operators of the form $(xD)^n$, $D^n x^n$, $(\Delta x)^n$, etc., are often difficult to work with and we seek equivalent forms. First we note that

$$(2.1) \qquad (xD)_n = A_n(xD) = \sum_{k=1}^{n} S(n,k)(xD)^k = x^n D^n$$

follows by indaction from

$$(xD)_{k+1} = (xD)_k(xD-k) = x^k D^k(xD-k) = x^k(D^k x)D - kx^k D^k$$
$$= x^k(xD^k + kD^{k-1})D - kx^k D^k = x^{k+1}D^{k+1} .$$

But (2.1) admits the inverse

$$(2.2) \qquad (xD)^n = \Sigma S(n,k) x^k D^k = B_n[xD] .$$

Equation (2.2) can slo be shown directly using the recurrence for $S(n,k)$ [4, p. 218].

Similarly,

$$(2.3) \qquad (x\Delta)_n = A_n(x\Delta) = \sum_{k=0}^{n} a(n,k)(x\Delta)^k = x^{(n)}\Delta^n$$

follows by induction from

$$(x\Delta)_{k+1} = (x\Delta-k)(x\Delta)_k = (x\Delta-k)x^{(k)}\Delta^k = \left\{x\Delta x^{(k)} - kx^{(k)}\right\}\Delta^k$$
$$= \left\{xx^{(k)}\Delta + kx(x+1)^{(k-1)} + kx(x+1)^{(k-1)}\Delta - kx^{(k)}\right\}\Delta^k$$
$$= \left\{xx^{(k)}\Delta + kx(x+1)^{k-1}\Delta\right\}\Delta^k = (x+k)x^{(k)}\Delta\Delta^k = x^{(k+1)}\Delta^{k+1} .$$

But (2.3) admits the inverse

$$(2.4) \qquad (x\Delta)^n = \Sigma S(n,k) x^{(k)}\Delta^k = B_n\left[x\Delta\right]$$

where $x^j \equiv x^{(j)}$.

Since

$$(Dx)^n = x^{-1}(xD)^{n+1}D^{-1} \qquad \text{and} \qquad (\Delta x)^n = x^{-1}(x\Delta)^{n+1}\Delta^{-1}$$

we have from (2.2) and (2.4), respectively,

$$(2.5) \qquad (Dx)^n = x^{-1}B_{n+1}[xD]D^{-1} = \sum_{k=1}^{n+1} S(n+1,k)x^{k-1}D^{k-1} ,$$

(2.6)
$$(\Delta x)^n = x^{-1}B_{n+1}[x\Delta]\Delta^{-1} = \sum_{k=1}^{n+1} S(n+1,k)(x+1)^{(k-1)}\Delta^{k-1}.$$

Using Leibnitz's formula for the derivative of a product we get; cf. [1. p.]

$$D^n x^n = \sum_{k=0}^{n} \binom{n}{k} (D^k x^n)D^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (n)_k x^{n-k} D^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} x^{n-k} D^{n-k}$$

Replacing $n-k$ by k we have

(2.7)
$$D^n x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} x^k D^k.$$

Using

$$D^{k+1}x^{k+1} = D^k\left\{x^{k+1}D + (k+1)x^k\right\} = D^k x^k\left\{xD + k + 1\right\}$$

we have by induction

(2.8)
$$D^n x^n = (xD+1)^{(n)} = (Dx)^{(n)} = A^{(n)}(Dx).$$

Since

$$(xD)^{(n)} = (xD)(xD+1)^{(n-1)} = (xD)(Dx)^{(n-1)} = xD\,D^{n-1}x^{n-1}$$

we have

$$(xD)^{(n)} = xD^n x^{n-1}.$$

Using the difference analogue of Leibnitz's formula [2, p. 96] we get cf. [1, p. 4],

$$\Delta^n x^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}E^k x^{(n)}\Delta^k = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k}(x+k)^{(n)}\Delta^k = \sum_{k=0}^{n} \binom{n}{k} (n)_{n-k}(x+n)^{(k)}\Delta^k.$$

Hence

(2.9)
$$\Delta^n x^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!}(x+n)^{(k)}\Delta^k.$$

Using

$$\begin{aligned}\Delta^{k+1}(x)_{k+1} &= \Delta^k(\Delta(x)_{k+1}) = \Delta^k\left\{(x)_{k+1}\Delta + (k+1)(x)_k + (k+1)(x)_k\Delta\right\}\\ &= \Delta^k(x)_k\left\{(x-k)\Delta + (k+1) + (k+1)\Delta\right\}\\ &= \Delta^k(x)_k(x\Delta + \Delta + 1 + k) = \Delta^k(x)_k(\Delta x + k),\end{aligned}$$

we have by induction

(2.10)
$$\Delta^n(x)_n = (\Delta x)^{(n)} = A^{(n)}(\Delta x).$$

But

$$\Delta^n x^{(n)} = \Delta^n(x+n-1)_n = (\Delta(x+n-1))^{(n)};$$

hence using $\Delta x = x\Delta + \Delta + 1$ we have

(2.11)
$$\Delta^n x^{(n)} = ((x+n)\Delta + 1)^{(n)} = ((x+n)\Delta + n)_n.$$

Taking the inverse of (2.8) we have

(2.12)
$$(Dx)^n = \sum_{k=1}^{n} (-1)^{n-k}S(n,k)D^k x^k = B^{(n)}[Dx].$$

Taking the inverse of (2.10) we have

$$(2.13)\qquad (\Delta x)^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)\Delta^k (x)_k = B^{(n)}[\Delta x] ,$$

where $x^j \equiv (x)_j$.

Since

$$(xD)^{m+n} = (xD)^m (xD)^n \qquad \text{and} \qquad \left\{(xD)^m\right\}^n = (xD)^{mn}$$

we have by (2.2)

$$(2.14)\qquad B_{m+n}[xD] = B_m[xD]B_n[xD] , \qquad (B_m[xD])^n = B_{mn}[xD] .$$

Similarly (2.4) gives

$$(2.15)\qquad B_{m+n}[x\Delta] = B_m[x\Delta]B_n[x\Delta], \qquad \left\{B_m[x\Delta]\right\}^n = B_{mn}[x\Delta] .$$

Similar results also hold for $B^{(k)}[Dx]$ and $B^{(k)}[\Delta x]$.

3. THE OPERATORS $x(I + D), x(I + \Delta), (I + D)x, (I + \Delta)x$

Analogous to (2.1) is

$$(3.1)\qquad (x(I+D))_n = A_n(x(I+D)) = x^n(I+D)^n = [x(I+D)]^n$$

which follows by induction from

$$(x(I+D))_{k+1} = (x(I+D))_k(x(I+D)-k) = x^k(I+D)^k(x(I+D)-k)$$
$$= x^k\left\{x(I+D)^{k+1} + k(I+D)^k - k(I+D)^k\right\} = x^{k+1}(I+D)^{k+1} .$$

But (3.1) admits the inverse

$$(3.2)\qquad (x(I+D))^n = \sum_{k=1}^{n} S(n,k)x^k(I+D)^k = B_n[x(I+D)] .$$

Since

$$((I+D)x)^n = x^{-1}(x(I+D))^{n+1}(I+D)^{-1}$$

we have

$$(3.3)\qquad ((I+D)x)^n = \sum_{k=1}^{n+1} S(n+1, k)x^{k-1}(I+D)^{k-1} .$$

Using

$$(I+D)^{n+1}x^{n+1} = (I+D)^n(I+D)x^{n+1} = (I+D)^n x^n(x+xD+n+1) = (I+D)^n x^n((I+D)x+n)$$

we have by induction

$$(3.4)\qquad (I+D)^n x^n = ((I+D)x)^{(n)} = A^{(n)}((I+D)x)$$

which admits the inverse

$$(3.5)\qquad ((I+D)x)^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(I+D)^k x^k = B^{(n)}[(I+D)x] .$$

By (3.4) and since $(I+D)x = (x+xD)+1$,

$$(x(I+D))^{(n)} = (x+xD)^{(n)} = x(I+D)((I+D)x)^{n-1} = x(I+D)(I+D)^{n-1}x^{n-1} .$$

Hence

(3.6) $$(x(I+D))^{(n)} = x(I+D)^n x^{n-1} .$$

By (3.1) and since

$$(x+Dx)_n = (x+Dx)(x+xD)_n$$

we have

(3.7) $$((I+D)x)_n = (I+D)x^n(I+D)^{n-1} .$$

Using (1.4)

(3.8) $$(x(I+\Delta))^{(n)} = \sum_{k=1}^{n} (-1)^{n-k}s(n,k)(x(I+\Delta))^k = A^{(n)}(x(I+\Delta)).$$

But,

(3.9) $$(x(I+\Delta))^n = x^{(n)}(I+\Delta)^n$$

follows by induction from

$$(x(I+\Delta))^{k+1} = (x(I+\Delta))(x(I+\Delta))^k = x(I+\Delta)x^{(k)}(I+\Delta)^k$$

$$= x\left\{x^{(k)} + x^{(k)}\Delta + k(x+1)^{(k-1)} + k(x+1)^{(k-1)}\Delta\right\}(I+\Delta)^k$$

$$= x\left\{x^{(k)} + k(x+1)^{(k-1)}\right\}(I+\Delta)^{k+1} = x(x+1)^{(k-1)}(x+k)(I+\Delta)^{k+1} = x^{(k+1)}(I+\Delta)^{k+1} .$$

Hence

(3.10) $$(x(I+\Delta))^{(n)} = \sum_{k=1}^{n} (-1)^{n-k}s(n,k)x^{(k)}(I+\Delta)^k = A^{(n)}[x(I+\Delta)] ,$$

where $x^k \equiv x^{(k)}$.

Relation (3.8) admits the inverse

(3.11) $$(x(I+\Delta))^n = \sum_{k=1}^{n} (-1)^{n-k}S(n,k)(x(I+\Delta))^{(k)} = B^{(n)}(x(I+\Delta)),$$

where $(x(I+\Delta))^k \equiv (x(I+\Delta))^{(k)}$.

Using (3.9), (3.11) may be rewritten

(3.12) $$(x)^{(n)}(I+\Delta)^n = \sum_{k=1}^{n} (-1)^{n-k}S(n,k)(x(I+\Delta))^{(k)} .$$

Using (1.1)

(3.13) $$(x(I+\Delta))_n = \sum_{k=1}^{n} s(n,k)(x(I+\Delta))^k = A_n(x(I+\Delta))$$

and using (3.9)

(3.14) $$(x(I+\Delta))_n = \sum_{k=1}^{n} s(n,k)x^{(k)}(I+\Delta)^k = A_n[x(I+\Delta)] ,$$

where the inverses of (3.13 and (3.14 are, respectively,

(3.15) $$(x(I+\Delta))^n = \sum_{k=1}^{n} S(n,k)(x(I+\Delta))_k = B_n(x(I+\Delta))$$

and

$$x^{(n)}(I+\Delta)^n = \sum_{k=1}^{n} S(n,k)(x(I+\Delta))_k = B_n(x(I+\Delta)). \tag{3.16}$$

Iterating $(I+\Delta)x = x + x\Delta + \Delta + I = (x+1)(I+\Delta)$ n times we have

$$(I+\Delta)^n x = (x+n)(I+\Delta)^n . \tag{3.17}$$

More generally,

$$(I+\Delta)^n x^{(n)} = (x+n)^{(n)}(I+\Delta)^n \tag{3.18}$$

as the following induction step shows:

$$(I+\Delta)^{n+1}x^{(n+1)} = (I+\Delta)^n(I+\Delta)x^{(n+1)} = (I+\Delta)^n(x+1)^{(n)}(x+n+1)(I+\Delta)$$

$$= (x+1+n)^{(n)}(I+\Delta)^n(x+n+1)(I+\Delta).$$

Using (3.17) we get

$$(x+1+n)^{(n)}(x+n+1+n)(I+\Delta)^n(I+\Delta) = (x+n+1)^{(n+1)}(I+\Delta)^{n+1} .$$

Replacing x by $x+1$ in (3.9) and using (3.17) for $n=1$ we have

$$((I+\Delta)x)^n = (x+1)^{(n)}(I+\Delta)^n = (I+\Delta)^n(x)_n . \tag{3.19}$$

Similarly (3.10) becomes

$$((I+\Delta)x)^{(n)} = A^{(n)}[(x+1)(I+\Delta)] = A^{(n)}[(I+\Delta)x], \tag{3.20}$$

where $(x+1)^k \equiv (x+1)^{(k)}$.

Equation (3.11) becomes

$$((I+\Delta)x)^n = B^{(n)}((x+1)(I+\Delta)) = B^{(n)}[(I+\Delta)x]. \tag{3.21}$$

Equation (3.14) becomes

$$((I+\Delta)x)_n = A_n[(I+\Delta)x]. \tag{3.22}$$

4. THE OPERATORS $xD^2x\ Dx^2D,\ x\Delta^2x - 1,\ \Delta(x-1)^{(2)}\Delta$

We first note that xD and Dx commute, i.e.,

$$xD^2x = xD\,Dx = x^2D^2 + 2xD = DxxD = Dx^2D \tag{4.1}$$

and we restrict our attention to xD^2x.

Since $xD^2x = xD\,Dx = xD(1+xD) = B_1[xD](1+B_1[xD])$,

$$(xD^2x)^n = \left\{B_1[xD](1+B_1[xD])\right\}^n .$$

By (2.14) this gives

$$(xD^2x)^n = B_n[xD](1+B_1[xD])^n \tag{4.2}$$

or alternatively

$$(xD^2x)^n = \sum_{k=0}^{n} \binom{n}{k} B_{n+k}[xD] . \tag{4.3}$$

This becomes

(4.4) $$(xD^2x)^n = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n+k} S(n+k,j)x^jD^j$$

or utilizing (2.2),

(4.5) $$(xD^2x)^n = \sum_{k=0}^{n} \binom{n}{k} (xD)^{n+k} .$$

Since xD and Dx commute with each other,

$$(xD^2x)^n = (xD\,Dx)^n = (xD)^n(Dx)^n = [(xD)(Dx)]^n .$$

Using (2.2) and (2.12) this gives

(4.6) $$(xD^2x)^n = B_n[xD]B^{(n)}[Dx]$$

Comparison with (4.2) yields

(4.7) $$B^{(n)}[Dx] = \sum_{k=0}^{n} \binom{n}{k} B_k[xD] .$$

Since by (2.1) and (2.8),

$$x^nD^{2n}x^n = x^nD^nD^nx^n = (xD)_n(Dx)^{(n)}$$

and since

$$(xD-k)(Dx+k) = (xD-k)(xD+1+k) = xD^2x-k^{(2)}$$

we have, analogous to (2.1) and (2.8),

(4.8) $$x^nD^{2n}x^n = \prod_{k=0}^{n} (xD^2x-k^{(2)}).$$

Remark. $$D^nx^{2n}D^n = x^nD^{2n}x^n .$$

We note that $x\Delta$ and $\Delta(x-1)$ commute, i.e.,

(4.9) $$x\Delta^2(x-1) = x\Delta(1+x\Delta) = (1+x\Delta)x = (x-1)^{(2)}\Delta .$$

Writing

$$x\Delta^2(x-1) = x\Delta(1+x\Delta) = B_1[x\Delta](1+B_1[x\Delta])$$

we have using (2.14)

(4.10) $$(x\Delta^2(x-1))^n = B_n[x\Delta](1+B[x\Delta])^n$$

or

(4.11) $$(x\Delta^2(x-1))^n = \sum_{k=0}^{n} \binom{n}{k} B_{n+k}[x\Delta] = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n+k} S(n+k,j)x^jD^j$$

or using (2.4)

(4.12) $$(x\Delta^2(x-1))^n = \sum_{k=0}^{n} \binom{n}{k} (x\Delta)^{n+k} .$$

Since by (2.3) and (2.10)

$$x^{(n)}\Delta^n\Delta^n(x-1)_n = (x\Delta)_n(\Delta(x-1))^{(n)} = (x\Delta)_n(x\Delta+1)^{(n)}$$

and since

$$(x\Delta-k)(x\Delta+1+k) = (x\Delta^2(x-1)-k^{(2)})$$

we have, analogous to (4.8),

(4.13) $$x^{(n)}\Delta^{2n}(x-1)_n = \prod_{k=0}^{n}(x\Delta^2(x-1)-k^{(2)}).$$

5. THE OPERATORS $x(I+D)^2x,\ x(I+\Delta)^2(x-1)$

The operators $x(I+D)$ and $(I+D)x$ commute, i.e.,

(5.1) $$x(I+D)^2x = (I+D)x^2(I+D),$$

and we have using (3.2)

(5.2) $$(x(I+D)^2x)^n = \sum_{k=0}^{n}\binom{n}{k}B_{n+k}[x(I+D)] = \sum_{k=0}^{n}\binom{n}{k}(x(I+D))^{n+k}$$

and

(5.3) $$(x(I+D)^2x)^n = \sum_{n=0}^{n}\binom{n}{k}\sum_{j=0}^{n+k}S(n+k,j)x^j(I+D)^j\ .$$

The operators $x(I+\Delta)$ and $(I+\Delta)(x-1)$ commute, i.e.,

(5.4) $$x(I+\Delta)^2(x-1) = (I+\Delta)(x-1)^{(2)}(I+\Delta).$$

Using (3.18),

(5.5) $$x(I+\Delta)^2(x-1) = x(I+\Delta)x(I+\Delta) = (x(I+\Delta))^2.$$

Hence by (3.9)

(5.6) $$x(I+\Delta)^2(x-1))^n = (x(I+\Delta))^{2n} = x^{(2n)}(1+\Delta)^{2n}\ .$$

Since

$$x^{(n)}(I+\Delta)^n(I+\Delta)^n(x-1)_n = x^{(n)}(1+\Delta)^n(1+\Delta)^n(x-n)^{(n)}$$
$$= x^{(n)}(1+\Delta)^n x^{(n)}(1+\Delta)^n = x^{(n)}(x+n)^{(n)}(I+\Delta)^n(1+\Delta)^n$$

we have

(5.7) $$x^{(n)}(1+\Delta)^{2n}(x-1)_n = x^{(2n)}(1+\Delta)^{2n}$$

and comparing with (5.6)

(5.8) $$(x(I+\Delta)^2(x-1))^n = x^{(n)}(I+\Delta)^{2n}(x-1)_n\ .$$

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