COLUMN GENERATORS FOR COEFFICIENTS OF FIBONACCI AND FIBONACCI-RELATED POLYNOMIALS

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1. INTRODUCTION

Generating functions, row sums and rising diagonal sums for the Pascal triangle and types of Pascal triangles have been studied in [2] and [4]. Bicknell has pointed out in [1] that another Pascal-like array is observed if we consider the coefficients of the Fibonacci polynomials $F_n(t)$. These polynomials are such that

$$F_0(t) = 0, \quad F_1(t) = 1, \quad \text{and} \quad F_n(t) = tF_{n-1}(t) + F_{n-2}(t)$$

for $n \geq 2$. The array is as follows:

Array 1

	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8
0	0								
1	1								
2	0	1							
3	1	0	1						
4	0	2	0	1					
5	1	0	3	0	1				
6	0	3	0	4	0	1			
7	1	0	6	0	5	0	1		
8	0	4	0	10	0	6	0	1	
9	1	0	10	0	15	0	7	0	1

$\frac{x}{1-x^2}$ $\quad \frac{x^2}{(1-x^2)^2}$ $\quad \frac{x^3}{(1-x^2)^3}$ $\quad \frac{x^4}{(1-x^2)^4}$ $\cdots$ Column Generators

0^{th} $\quad 1^{st}$ $\quad 2^{nd}$ $\quad 3^{rd}$ Column

Since the generating function for the zeroth column is $f(x) = x/(1-x^2)$ and since each nonzero a_{ij} has the Pascal-like property

$$a_{ij} = \sum_{k=0}^{i-1} a_{k,j-1}$$

for all i and j such that $i > j \geq 1$, then techniques similar to those in Theorem 1 of [4] can be used to show that the generating function for the k^{th} column ($k = 0, 1, 2, \cdots$) is

$$g_k(x) = f(x)[x/(1-x^2)]^k.$$

Moreover, the generating function for the row sums of this array is

$$G(x) = \sum_{k=0}^{\infty} g_k(x) = f(x) \sum_{k=0}^{\infty} \left(\frac{x}{1-x^2}\right)^k = f(x)\,\frac{1-x^2}{1-x-x^2} = \frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n$$

as was to be expected. Again, employing results essentially the same as those in [2] and [4], the generating function for the rising diagonals of this array is

$$D(x) = \sum_{k=0}^{\infty} x^k g_k(x) = f(x) \sum_{k=0}^{\infty} \left(\frac{x^2}{1-x^2}\right)^k = \left(\frac{x}{1-x^2}\right)\left(\frac{1-x^2}{1-2x^2}\right) = \frac{x}{1-2x^2} = \sum_{n=0}^{\infty} 2^n x^{2n+1}.$$

2. GENERATING FUNCTIONS FOR COEFFICIENTS OF $F_n'(t)$

Now we consider the array for $F_n'(t)$, the first derivative of each Fibonacci polynomial. It will be noted that this array is quite similar to the array suggested by Hoggatt in problem H-131 of this *Quarterly* [5]. In that problem it is required to show that sums, C_n, of the rising diagonals are given by $C_1 = 0$ and

$$C_{n+1} = \sum_{j=0}^{n} F_{n-j}F_j .$$

If we appropriately relabel the columns in that array, this is the same as showing

$$C_n = \sum_{j=0}^{n} F_{n-j}F_j$$

for $n = 0, 1, 2, \cdots$. Since the rising diagonal sums of that array are the same as the row sums of the array for the $F_n'(t)$ and since we can find the column generators for the array below, we can employ techniques similar to those used in the previous section to answer problem H-131. For consider:

Array 2

n	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7
0	0							
1	0							
2	1							
3	0	2						
4	2	0	3					
5	0	6	0	4				
6	3	0	12	0	5			
7	0	12	0	20	0	6		
8	4	0	30	0	30	0	7	
9	0	20	0	30	0	42	0	8
				...				
	$\frac{x^2}{(1-x^2)^2}$	$\frac{2x^3}{(1-x^2)^3}$	$\frac{3x^4}{(1-x^2)^4}$	$\frac{4x^5}{(1-x^2)^5}$		Column Generators		
	0th	1st	2nd	3rd		Column		

Denoting the generator of the zeroth column as $p(x)$, the column generator for the k^{th} column is given by

$$d_k(x) = p(x)(k+1)\left(\frac{x}{1-x^2}\right)^k$$

for $k = 0, 1, 2, \cdots$. The generating function for the row sums is given by

$$G(x) = \sum_{k=0}^{\infty} d_k(x) = p(x) \sum_{k=0}^{\infty} (k+1)\left(\frac{x}{1-x^2}\right)^k = p(x)\,\frac{1}{\left[1-\frac{x}{1-x^2}\right]^2}$$

$$= \frac{x^2}{(1-x^2)} \cdot \frac{(1-x^2)^2}{(1-x-x^2)^2} = \left(\frac{x}{1-x-x^2}\right)^2$$

$$= \sum_{n=0}^{\infty} F_n(x) \cdot \sum_{n=0}^{\infty} F_n(x) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} F_{n-j}F_j\right)x^n .$$

Since we have relabeled the zero$^{\text{th}}$ column, we have immediately that

$$C_n = \sum_{j=0}^{n} F_{n-j}F_j = F_n^{(1)}$$

for $n = 0, 1, 2, \cdots$, where $F_n^{(1)}$ represents the first Fibonacci convolution sequence [3].

3. GENERATING FUNCTIONS FOR THE COEFFICIENTS OF $I_n(t) = n \int_0^t F_n(t)dt$

The preceding suggests it would be in order to consider the array for

$$\int_0^t F_n(t)dt.$$

But this leads to an array containing fractions. To avoid this situation we consider the array for

$$I_n(t) = n \int_0^t F_n(t)dt$$

instead. This array now follows:

Array 3

n	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9
0	0									
1	0	1								
2	0	0	1							
3	0	3	0	1						
4	0	0	4	0	1					
5	0	5	0	5	0	1				
6	0	0	9	0	6	0	1			
7	0	7	0	14	0	7	0	1		
8	0	0	16	0	20	0	8	0	1	
9	0	9	0	30	0	27	0	9	0	1

This array looks familiar since the array for the Lucas polynomials is Array 4 at the top of the next page.

It is easy to establish that

$$L_{2k-1}(t) = (2k-1)\int_0^t F_{2k-1}(t)dt = I_{2k-1}(t)$$

and

Array 4

n	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9
0	2									
1	0	1								
2	2	0	1							
3	0	3	0	1						
4	2	0	4	0	1					
5	0	5	0	5	0	1				
6	2	0	9	0	6	0	1			
7	0	7	0	14	0	7	0	1		
8	2	0	16	0	20	0	8	0	1	
9	0	9	0	30	0	27	0	9	0	1

...

$$L_{2k}(t) = (2k)\int_0^t F_{2k}(t)dt + 2 = I_{2k}(t) + 2$$

for $k = 1, 2, 3, \cdots$; or if you prefer,

$$D_t[L_n(t)] = nF_n(t)$$

for $n = 1, 2, 3, \cdots$.

It is interesting to note that in each of the above arrays if we consider the left-most column as the zero$^{\text{th}}$ column, we do not obtain a Pascal-like triangle. However, if we consider the next column over as the zero$^{\text{th}}$ column, then we do have a Pascal-like array and the results of [4] are applicable.

Array 5

n	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9
0	0								
1	1								
2	0	1							
3	3	0	1						
4	0	4	0	1					
5	5	0	5	0	1				
6	0	9	0	6	0	1			
7	7	0	14	0	7	0	1		
8	0	16	0	20	0	8	0	1	
9	9	0	30	0	27	0	9	0	1

...

$\frac{x(1+x^2)}{(1-x^2)^2}$	$\frac{x^2+x^4}{(1-x^2)^3}$	$\frac{x^3+x^5}{(1-x^2)^4}$	$\frac{x^4+x^6}{(1-x^2)^5}$	$\frac{x^5+x^7}{(1-x^2)^5}$	Column Generators
0^{th}	1^{st}	2^{nd}	3^{rd}	4^{th}	Column

If we denote the generator of the 0^{th} column by $q(x)$, then the column generator for the k^{th} column $(k = 0,1,2,\cdots)$ is

$$h_k(x) = q(x)[x/(1-x^2)]^k.$$

The generating function for the row sums is

$$G(x) = \sum_{k=0}^{\infty} h_k(x) = q(x)\,\frac{1-x^2}{(1-x-x^2)} = (1+x^2)\left(\frac{x}{1-x^2}\right)\left(\frac{1}{1-x-x^2}\right).$$

The generating function for the rising diagonals is

$$D(x) = \sum_{k=0}^{\infty} x^k h_k(x) = q(x)\cdot\frac{1-x^2}{1-2x^2} = (1+x^2)\left(\frac{x}{1-x^2}\right)\left(\frac{1}{1-2x^2}\right) \quad .$$

4. RELATIONSHIPS AMONG THE GENERATING FUNCTIONS

We now observe some relationships between the generating functions $g_k(x)$, $d_k(x)$ and $h_k(x)$. First

$$d_k(x) = (k+1)xg_k(x)$$

which was to have been anticipated in light of the connection between Array 2 and Problem H-131. However, the relationship between $h_k(x)$ and $g_k(x)$ is a little more surprising. Since Array 5 was obtained via an integration process, it might be felt that $h_k(x)$ should relate in some way to an integral of $g_k(x)$; but

$$h_k(x) = \frac{x}{(k+1)}\, g_k'(x)$$

which is easy to verify. This formula can be used to investigate some integral relationships however. Assuming each function is defined on $[o,x]$ and using an integration-by-parts formula we have

$$(k+1)\int_0^x h_k(x)dx + \int_o^x g_k(x)dx = xg_k(x).$$

Since $d_k(x) = (k+1)xg_k(x)$, we now have

$$d_k(x) = (k+1)^2 \int_o^x h_k(x)dx + (k+1)\int_o^x g_k(x)dx\,,$$

a formula involving all three generating functions for $k = 0, 1, 2, 3, \cdots$.

REFERENCES

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3. V. E. Hoggatt, Jr., and Joseph Arkin, "A Bouquet of Convolutions," *The Fibonacci Quarterly,* to appear.
4. Stephen W. Smith and Dean B. Priest, "Row and Rising Diagonal Sums for a Type of Pascal Triangle," *The Fibonacci Quarterly,*
5. V. E. Hoggatt, Jr., Problem H-131, *The Fibonacci Quarterly,* Vol. 7, No. 3 (Oct. 1969), p. 285.

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