LUCAS POLYNOMIALS AND CERTAIN CIRCULAR FUNCTIONS OF MATRICES

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INTRODUCTION

1. The *fundamental function* $U_n(p,q)$ as defined by Lucas [4] uses the second-order recurrence relation

(1) $$U_{n+2} = pU_{n+1} - qU_n \qquad (n \geqslant 0)$$

with initial values $U_0 = 0$ and $U_1 = 1$. For example, we find by calculation, that

(1′) $$\begin{cases} U_2 = p & U_3 = p^2 - q \\ U_4 = p^3 - 2pq & U_5 = p^4 - 3p^2q + q^2 \cdots \end{cases}$$

so that, by induction

(2) $$U_n = \sum_{r=0}^{[n/2]} (-1)^r \binom{n-r}{r} p^{n-2r}q^r$$

As the sequence $\{U_n\}$ has only been defined for $n \geqslant 0$, and as we often require negative-valued subscripts, we find, by calculation of the U's that

(3) $$U_{-n} = -q^{-n}U_n$$

to allow unrestricted values of n.

2. In addition, Lucas [4] also defined the *primordial function* $V_n(p,q)$ by

(4) $$V_{n+2} = pV_{n+1} - qV_n \qquad (n \geqslant 0)$$

with $V_0 = 2$ and $V_1 = p$. For example,

(4′) $$\begin{cases} V_2 = p^2 - 2q & V_3 = p^3 - 3pq \\ V_4 = p^4 - 4p^2q + 2q^2 & V_5 = p^5 - 5p^3q + 5pq^2 \end{cases}$$
$$\cdots$$

As in Lucas [4], it can easily be verified that

(5) $$V_{2n+1} = pU_{2n+1} - 2qU_{2n}$$

and

(6) $$V_{2n+1} = 2U_{2n+2} - pU_{2n+1}\ .$$

3 In [1], Barakat considered the matrix exponential e^X for the 2×2 matrix

(7) $$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} ,$$

where he took

(8) $$trX = p \qquad \text{and} \qquad \det X = q\,.$$

By showing that we could express X^n in terms of the U_n for unrestricted values of n, viz:

(9) $$X^n = U_nX - qU_{n-1}I \qquad \text{and} \qquad X^{-n} = -qU_{-n}X^{-1} + U_{-n+1}I$$

(where I is the unit matrix of order 2).

Barakat [1] was then able to obtain various summation formulas for the Lucas polynomials by the use of the matrix exponential function, where

(10) $$e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad \text{and} \qquad e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} X^{-n}$$

4. It is the purpose of this paper to extend the work of Barakat [1] by considering the matrix sine and cosine for 2×2 matrices, and their corresponding connections with the sequences $\{U_n\}$ and $\{V_n\}$. As special cases, we will then examine the relationships between the Lucas polynomials and the Chebychev polynomials. We commence with an investigation of the sine of a matrix. For every square matrix X, the sine of X is defined by the power series

(11) $$\sin X = \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n+1}}{(2n+1)!}$$

We then give a set of parallel results for the cosine function, where we define the cosine of every square matrix X by the power series

(12) $$\cos X = \sum_{n=0}^{\infty} (-1)^n \frac{X^{2n}}{(2n)!}\,.$$

Expansions (11) and (12) are perfectly valid since, as the functions $\sin z$ and $\cos z$ converge for all z, the eigenvalues of X lie within the circle of convergence of radius $R = \infty$.

Summation Formulas – The Sine

5. If we substitute (9) into (11), then

$$\sin X = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (U_{2n+1}X - qU_{2n}I)$$

Thus, we have

(13) $$\sin X = X \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} U_{2n+1} - Iq \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} U_{2n}$$

6 By using Sylvester's matrix interpolation formula, viz. Bellman [2]:

If $f(t)$ is a polynomial of degree $\leqslant N-1$, and if $\lambda_1, \lambda_2, \cdots \lambda_N$ are the N distinct eigenvalues of X, then

(14) $$f(X) = \sum_{i=1}^{N} f(\lambda_i) \prod_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} \left[\frac{X - \lambda_j I}{\lambda_i - \lambda_j}\right],$$

we can show that if λ_1 and λ_2 are the eigenvalues of our 2×2 matrix X defined in (7), then

$$f(X) = \sum_{i=1}^{2} f(\lambda_i) \prod_{\substack{1 \leqslant j \leqslant N \\ j \neq 1}} \left[\frac{X - \lambda_j I}{\lambda_i - \lambda_j}\right] = f(\lambda_1) \prod_{\substack{1 \leqslant j \leqslant N \\ j \neq 1}} \left[\frac{X - \lambda_1 I}{\lambda_1 - \lambda_j}\right] + f(\lambda_2) \prod_{\substack{1 \leqslant j \leqslant N \\ j \neq 2}} \left[\frac{X - \lambda_2 I}{\lambda_2 - \lambda_j}\right] =$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left\{(X - \lambda_1 I)f(\lambda_1)\right\} - \frac{1}{\lambda_1 - \lambda_2}\left\{(X - \lambda_2 I)f(\lambda_2)\right\}$$

Hence, we have

$$\sin X = \frac{1}{\lambda_1 - \lambda_2}\left\{(X - \lambda_2 I)\sin\lambda_1 - (X - \lambda_2 I)\sin\lambda_2\right\}$$

so that

(15) $$\sin X = \frac{1}{\lambda_1 - \lambda_2}\left[(\sin\lambda_1 - \sin\lambda_2)X - (\lambda_1\sin\lambda_1 - \lambda_2\sin\lambda_2)I\right]$$

7. Now, the characteristic equation of X is

$$|X - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= a_{11}a_{22} - \lambda(a_{11} + a_{22}) + \lambda^2 - a_{12}a_{21}$$

$$= \lambda^2 - p\lambda + q = 0\,.$$

Thus, as in Barakat [1], the eigenvalues λ_1 and λ_2 satisfy the quadratic equation

(16) $$\lambda^2 - p\lambda + q = 0$$

so that

(17) $$\lambda_1 = \frac{p+\delta}{2} \quad\text{and}\quad \lambda_2 = \frac{p-\delta}{2} \quad\text{(say)}$$

(18) $$\delta = \Delta^{1/2} = (p^2 - 4q)^{1/2}\,.$$

8. Substituting these values for λ_1 and λ_2 in (15) eventually gives

(19) $$\sin X = \left[2\delta^{-1}\sin\frac{\delta}{2}\cos\frac{p}{2}\right]X - \left[\delta^{-1}p\sin\frac{\delta}{2}\cos\frac{p}{2} + \sin\frac{p}{2}\cos\frac{\delta}{2}\right]I\,.$$

Thus, on comparing Eqs. (13) and (19), we see that

(20) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+1} = 2\delta^{-1}\sin\frac{\delta}{2}\cos\frac{p}{2}$$

and

(21) $$q\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n} = \delta^{-1}p\sin\frac{\delta}{2}\cos\frac{p}{2} + \sin\frac{p}{2}\cos\frac{\delta}{2}\,.$$

9. If we rewrite (5) in the form

(22) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}V_{2n+1} = p\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+1} - 2q\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n}$$

we have, on using (20) and (21), that

(23) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}V_{2n+1} = -2\sin\frac{p}{2}\cos\frac{\delta}{2}\,.$$

Re-writing (6) as

(24) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}V_{2n+1} = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+2} - p\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+1}$$

gives

(25) $$2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}V_{2n+1} + p\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}U_{2n+1}$$

Using (20) and (24) in (25) yields, on calculation,

(26) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} U_{2n+2} = \delta^{-1} p \sin\frac{\delta}{2}\cos\frac{p}{2} - \sin\frac{p}{2}\cos\frac{\delta}{2}.$$

Summation Formulas – The Cosine

10. If we parallel the work in paragraphs 5 to 9 for the cosine of the matrix X as defined in (12), we also have the following results:

(27) $$\cos X = I - X \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n} + Iq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n-1}$$

so that

(28) $$\cos X = -X \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n} + Iq \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n-1}$$

since, when $n = 0$,

$$-X(-1)U_0 = 0$$

on using (1) and

$$IqU_{-1} = Iq \cdot q^{-1} = I$$

on using (3).

(29) $$\cos X = \left[-2\delta^{-1} \sin\frac{p}{2}\sin\frac{\delta}{2}\right] X - \left[\cos\frac{p}{2}\cos\frac{\delta}{2} - \delta^{-1} p \sin\frac{p}{2}\sin\frac{\delta}{2}\right] I$$

(30) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n} = 2\delta^{-1} \sin\frac{p}{2}\sin\frac{\delta}{2}$$

(31) $$q \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n-1} = \delta^{-1} p \sin\frac{p}{2}\sin\frac{\delta}{2} - \cos\frac{p}{2}\cos\frac{\delta}{2}$$

(32) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} V_{2n} = 2\cos\frac{p}{2}\cos\frac{\delta}{2}$$

(33) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} U_{2n+1} = \delta^{-1} p \sin\frac{p}{2}\sin\frac{\delta}{2} + \cos\frac{p}{2}\cos\frac{\delta}{2}$$

Chebychev Polynomials

11. As in Horadam [3], which deals among other things with Chebychev polynomials in relation to a certain generalized recurrence sequence, write

(34) $$x = \cos\theta \quad \text{with} \quad p = 2x \quad \text{and} \quad q = 1.$$

Then the U_n are precisely the Chebychev polynomials of the first kind, $S_n(x)$. Thus

(35) $$U_n(2x, 1) = S_n(x) = \frac{\sin n\theta}{\sin\theta} \qquad (n \geqslant 0),$$

where

(36) $$S_{n+2} = 2xS_{n+1} - S_n \quad \text{with} \quad S_0 = 0 \quad \text{and} \quad S_1 = 1.$$

Likewise, the V_n are the Chebychev polynomials of the second kind, $t_n(x) = 2T_n(x)$, where

(37) $$T_{n+2} = 2xT_{n+1} - T_n \quad \text{with} \quad T_0 = 1 \quad \text{and} \quad T_1 = x$$

so that

(38) $$t_0 = 2 \qquad \text{and} \qquad t_1 = 2x\,(=p).$$

Thus

(39) $$V_n(2x, 1) = 2T_n(x) = 2\cos n\theta \quad (n \geqslant 0).$$

Putting $q = 1$ in (20) and using (35) yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} U_{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} S_{2n+1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \sin(2n+1)x}{(2n+1)!\sin x}$$

$$= \frac{1}{\sin X} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left\{ \frac{e^{i(2n+1)x} - e^{-i(2n+1)x}}{2i} \right\}$$

$$= \frac{1}{\sin X} \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n (e^{ix})^{2n+1}}{(2n+1)!} - \frac{(-1)^n (e^{-ix})^{2n+1}}{(2n+1)!} \right\}$$

$$= \frac{1}{2i\sin x} \left\{ \sin e^{ix} - \sin e^{-ix} \right\}$$

$$= \frac{1}{2i\sin x}\, 2\cos\frac{e^{ix}+e^{-ix}}{2}\, \sin\frac{e^{ix}-e^{-ix}}{2}$$

$$= \frac{1}{i\sin x} \cos(\cos x)\sin(i\sin x)$$

$$= \frac{1}{i\sin x} \cos(\cos x) i \sinh(\sin x)$$

$$= \frac{\cos(\cos x)\sinh(\sin x)}{\sin x}.$$

Thus, we have

(40) $$\sum_{n=0}^{\infty} \frac{(-1)^n \sin(2n+1)x}{(2n+1)!\sin x} = \frac{\cos(\cos x)\sinh(\sin x)}{\sin x}.$$

Similarly, from (21), (30) and (32) and using (35) and (37), it can be shown that

(41) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cos(2n+1)x = \sin(\cos x)\cosh(\sin x)$$

(42) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \sin 2nx}{(2n)!\sin x} = \frac{\sin(\cos x)\sinh(\sin x)}{\sin x}$$

and

(43) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cos 2nx}{(2n)!} = -\cos(\cos x)\cosh(\sin x)$$

REFERENCES

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4. E. Lucas, *Théorie des nombres*, Albert Blanchard, Paris, 1961, Ch. 18.

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