A METHOD FOR THE EVALUATION OF CERTAIN SUMS INVOLVING BINOMIAL COEFFICIENTS

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Recently T. V. Narayana presented two verifications of the sum

$$(1)\qquad S = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1}\, u^r v^s = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} C_{r_1s} u^r v^s$$

$$= \tfrac{1}{2}\left(1 - u - v - \sqrt{1 - 2(u+v) + (u-v)^2}\,\right)$$

first derived by him in [1], and by Kreweras in [2], [3]. No direct proof of this formula seems to have been given. It is the purpose of this note to present an analytic derivation of Eq. (1) and to suggest a method more generally applicable to summing series with binomial coefficients. The method involves the introduction of an integral representation for at least one of the binomial coefficients.

To begin with let us transform the series of Eq. (1) by using the integral representation

$$(2)\qquad \frac{1}{r+s-1} = \int_1^{\infty} \frac{dt}{t^{r+s}}$$

and interchange the orders of summation and integration (a step that can be justified in detail for values of u and v for which the original series converges). Then we can write

$$(3)\qquad S = \int_1^{\infty} dt \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \binom{r+s-1}{r}\binom{r+s-1}{s}\left(\frac{u}{t}\right)^r\left(\frac{v}{t}\right)^s$$

so that we need only find the sum of the simpler series

$$(4)\qquad F(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \binom{r+s-1}{r}\binom{r+s-1}{s} x^r y^s$$

with $x = u/t,\ y = v/t$. At this point we introduce the integral representation

$$(5)\qquad \binom{r+s-1}{s} = \frac{1}{2\pi i}\oint \frac{(1+z)^{r+s-1}}{z^{s+1}}\, dz\,,$$

where the contour will be chosen as the unit circle. We can again interchange orders of summation and integration to find

$$(6)\qquad F(x,y) = \frac{1}{2\pi i}\oint \sum_{r=1}^{\infty} x^r \sum_{s=1}^{\infty} \binom{r+s-1}{r} \frac{(1+z)^{r+s-1}}{z^{s+1}}\, y^s dz\,.$$

But the summation over s can be effected explicitly using the formula

$$\sum_{j=0}^{\infty} \binom{r+j}{j} a^j = \frac{1}{(1-a)^{r+1}} \tag{7}$$

valid for $|a| < 1$. In this way we find

$$\begin{aligned} F(x,y) &= \frac{y}{2\pi i} \oint \frac{dz}{z^2} \sum_{r=1}^{\infty} x^r (1+z)^r \frac{z^{r+1}}{[z-(1+z)y]^{r+1}} \\ &= -\frac{xy}{2\pi i} \oint \frac{dz(1+z)}{[z(1-y)-y][xz^2+z(x+y-1)+y]} \\ &= \frac{-y}{2\pi i(1-y)} \oint \frac{dz(1+z)}{\left(z-\frac{y}{1-y}\right)\left(z^2+z\left(\frac{x+y-1}{x}\right)+\frac{y}{x}\right)} . \end{aligned} \tag{8}$$

The quadratic form in z can be factored in the form

$$z^2 + z\left(\frac{x+y-1}{x}\right) + \frac{y}{x} = (z-z_+)(z-z_-) , \tag{9}$$

where

$$z_{\pm} = \frac{1}{2x}\left(1-x-y \pm \sqrt{(1-x-y)^2-4xy}\right) . \tag{10}$$

It is easily verified that the only root of Eq. (8) that lies in the unit circle as x or y tends to zero is z_-, hence in the evaluation of the contour integral in Eq. (8), we need only be concerned about the poles at $z = y/(1-y)$ and at $z = z_-$. The residue of the integrand at $z = y/(1-y)$ is found to be $(1-y)/y$ and the residue at $z = z_-$ is

$$\frac{1+z_-}{(z_--z_+)\left(z_--\frac{y}{1-y}\right)} = \frac{-(1-y)(1-x-y+\sqrt{(1-x-y)^2-4xy})}{2y\sqrt{(1-x-y)^2-4xy}} . \tag{11}$$

If we add the contributions from the two poles we find

$$F(x,y) = \frac{(1-x-y+\sqrt{(1-x-y)^2-4xy})}{2\sqrt{(1-x-y)^2-4xy}} - 1 = \frac{1-x-y-\sqrt{(1-x-y)^2-4xy}}{2\sqrt{(1-x-y)^2-4xy}} . \tag{12}$$

If we now return to the integral over t, we find that S can be expressed as

$$S = \int_1^{\infty} F(u/t, v/t)dt = \int_1^{\infty} \frac{t-u-v-\sqrt{(t-u-v)^2-4uv}}{2\sqrt{(t-u-v)^2-4uv}} dt . \tag{13}$$

Letting $t-u-v = \zeta$, we can transform this last integral to

$$S = \int_{1-u-v}^{\infty} \frac{\zeta-\sqrt{\zeta^2-4uv}}{2\sqrt{\zeta^2-4uv}} d\zeta . \tag{14}$$

Finally, the substitution $\zeta = 2\sqrt{uv}\cosh\theta$ allows us to express S as

$$\begin{aligned} S &= \sqrt{uv} \int_{\cosh^{-1}\left(\frac{1-u-v}{2\sqrt{uv}}\right)}^{\infty} e^{-\theta} d\theta = \sqrt{uv}\exp\left[-\cosh^{-1}\left(\frac{1-u-v}{2\sqrt{uv}}\right)\right] \\ &= \tfrac{1}{2}(1-u-v-\sqrt{(1-u-v)^2-4uv}) \end{aligned}$$

as found in the earlier references.

Another set of identities that has been the subject of several recent notes, [4]–[6], is the following

$$
\text{(16)}\qquad
\begin{aligned}
A &= \sum_{n=0}^{N} (-1)^n \binom{n+\epsilon-1}{n}\binom{\epsilon}{N-n} = 0\\
B &= \sum_{n=0}^{N} (-1)^n \binom{n+\epsilon-1}{N-1}\binom{N}{n} = 0
\end{aligned}
$$

These can both be derived in the same way as the identity of Eq. (1). In the expression for A we note that the upper limit of the sum can be chosen to be ∞ if we use the convention that

$$\binom{a}{-j} = 0$$

for j any positive integer. If we then use an integral representation for $\binom{\epsilon}{N-n}$ we find

$$\text{(17)}\qquad A = \frac{1}{2\pi i}\sum_{n=0}^{\infty} (-1)^n \binom{n+\epsilon-1}{n} \oint \frac{(1+z)^{\epsilon}}{z^{N+1-n}}\,dz = \frac{1}{2\pi i}\oint \frac{dz}{z^{N+1}} = 0 \quad .$$

Similarly the series of B can be expressed as

$$\text{(18)}\qquad B = \frac{1}{2\pi i}\sum_{n=0}^{N} (-1)^n \binom{N}{n} \oint \frac{(1+z)^{n+\epsilon-1}}{z^{N}}\,dz = \frac{(-1)^N}{2\pi i}\oint (1+z)^{\epsilon-1}dz = 0\,.$$

where the contour can be suitably modified when a branch cut must be made.

The preceeding analysis is of interest not for its derivation of known results but because it gives a method that can be tried on many similar problems. In cases where a summation in closed form is not possible, the integral representation can sometimes lead to asymptotic results.

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