ON A THEOREM OF KRONECKER

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Consider the r^{th} order homogeneous linear recursion

(1) $$u_{n+r} = a_1 u_{n+r-1} + \cdots + a_r u_n,\ a_r \neq 0,$$

over a field F of characteristic $p > 0$. Let

$$V = \left\{ v_n \right\}_o^\infty \subseteq F$$

be a non-trivial solution of the recursion (1) and let

$$f(x) \equiv x^r - a_1 x^{r-1} - \cdots - a_r = \prod_{i=1}^{r} (x - r_i)$$

be factored completely in its splitting field K where $F \subseteq K$. The results which follow remain valid if F is also the complex field. The polynomial $f(x)$ is called a characteristic polynomial for the sequence V. If $\phi(n)$ is a sequence in K defined on the non-negative integers I then define the operator E by $E\phi(n) \equiv \phi(n+1)$ for $n \in I$. Recursion (1) may therefore be written as

(2) $$f(E)u_n \equiv 0.$$

The sequence V is said to satisfy a recursion of lower order if there exists a monic polynomial $g(x)$ over F such that $\deg g(x) < \deg f(x) = r$ and $g(E)v_n \equiv 0$. There exists a unique monic polynomial of lowest degree which is a characteristic polynomial for V, called the minimum polynomial for V [2]. The determination of the lowest order recursion that a given solution of (2) satisfies is an essential step in the study of the periodicity properties of such solutions. Define

(3) $$D(n) = \det \begin{bmatrix} v_n & v_{n+1} & \cdots & v_{n+r-1} \\ v_{n+1} & v_{n+2} & & \\ \vdots & & \ddots & \vdots \\ v_{n+r-1} & \cdots & & v_{n+2r-2} \end{bmatrix},\quad n \in I.$$

The purpose of this note is to present a new proof of a classic theorem of Kronecker [1, p. 199] which does not depend on the notion of a fundamental solution set for (2). To this end Lemma 2 gives an explicit calculation of the values of $D(n)$.

Theorem 1. (Kronecker) The solution V of (2) satisfies a recursion of lower order if and only if $D(0) = 0$.

First define the polynomials

(4) $$f_k(x) = \prod_{i \neq k} (x - r_i),\qquad 1 \leqslant k \leqslant r.$$

We have

Lemma 1. $$f_k(E)v_n = r_k^n [f_k(E)v_o],\qquad n \in I.$$

Proof. Note that $f(x) = (x - r_k)f_k(x)$ and since polynomials in E commute as operators we have

(5) $$r_k f_k(E)v_n = E[f_k(E)v_n] = f_k(E)v_{n+1}.$$

The result follows from a repeated application of (5). Q.E.D.

Corollary 1. If $f_k(E)v_0 = 0$ then $f_k(E)v_n = 0,\ n \in I.$

The main result of this note is

Lemma 2. $$D(n) = (-1)^t \prod_{i=1}^{r} [f_i(E)v_n], \qquad t = r(r-1)/2, \quad n \in I.$$

Proof. (Induct on the order r of the recursion) If $r = 2$ then

$$D(n) = \det\begin{bmatrix} v_n & v_{n+1} \\ v_{n+1} & v_{n+2}\end{bmatrix} = v_nv_{n+2} - v_{n+1}^2 = -v_{n+1}^2 + v_n[(r_1+r_2)v_{n+1} - r_1r_2v_n]$$
$$= -(v_{n+1} - r_1v_n)(v_{n+1} - r_2v_n) = -[f_2(E)v_n][f_1(E)v_n].$$

Therefore the lemma is true for $r = 2$. Assume the lemma true for all recursions of order less than $r > 2$. Since

$$f_1(E)v_n = v_{n+r-1} + \sum_{i=2}^{r} c_iv_{n+r-i}$$

for some $c_i \in K$, we have that c_i times the $r+1-i$ row of $D(n)$ added to the r^{th} row for $2 \le i \le r$ gives

$$D(n) = \det\begin{bmatrix} v_n & v_{n+1} & \cdots & v_{n+r-1} \\ v_{n+1} & & & \\ \vdots & \vdots & & \vdots \\ v_{n+r-2} & & & \\ f_1(E)v_n & f_1(E)v_{n+1} & \cdots & f_1(E)v_{n+r-1}\end{bmatrix}$$

which, by Lemma 1, gives

$$D(n) = [f_1(E)v_n]\det\begin{bmatrix} v_n & v_{n+1} & \cdots & v_{n+r-1} \\ v_{n+1} & & & \\ \vdots & \vdots & & \vdots \\ v_{n+r-2} & & & \\ 1 & r_1 & & r_1^{r-1}\end{bmatrix}.$$

Multiplying column i by $-r_1$ and adding to column $i+1$ for $1 \le i \le r-1$, we have

(6) $$D(n) = (-1)^{r-1}[f_1(E)v_n]\det\begin{bmatrix} w_n & \cdots & w_{n+r-2} \\ w_{n+1} & \ddots & \vdots \\ \vdots & & \\ w_{n+r-2} & \cdots & w_{n+2r-4}\end{bmatrix}$$

for

$$w_n \equiv v_{n+1} - r_1v_n, \qquad n \in I,$$

where the matrix appearing in (6) is $r-1$ square. Note that

$$f_1(E)w_n = f(E)v_n = 0, \qquad n \in I,$$

so that $f_1(x)$ is a characteristic polynomial for the sequence $\{w_n\}$. Let

$$g_k(x) \equiv \prod_{i \ne 1,k} (x - r_i), \qquad 2 \le k \le r.$$

Then, by the induction hypothesis, Eq. (6) becomes

$$D(n) = (-1)^{r-1}[f_1(E)v_n](-1)^{(r-1)(r-2)/2}\prod_{i=2}^{r}[g_i(E)w_n] = (-1)^{r(r-1)/2}[f_1(E)v_n]\prod_{i=2}^{r}[f_i(E)v_n].$$

Therefore mathematical induction yields the result. Q.E.D.

An immediate consequence of Corollary 1 and Lemma 2 is

Corollary 2. Either $D(n)$ is identically zero or never zero.

Zierler proves the following [2].

Lemma 3. Let $f(x)$ be a characteristic polynomial over the field F for the sequence

$$V = \{ v_n \} \subseteq F, \qquad V \neq 0,$$

and let $g(x)$ be the minimum polynomial for V. Then

(i) $$g(x) \mid f(x),$$

(ii) $h(x)g(x)$ is also a characteristic polynomial for V, where $h(x)$ is any monic polynomial over F.

To complete the proof of Theorem 1 we note that Lemma 3 implies that V satisfies a lower order recursion if and only if some $f_k(x)$ as defined in (4) is a characteristic polynomial for V. But then Lemma 2 and Corollary 2 imply that V satisfies a lower order recursion if and only if $D(0) = 0$.

REFERENCES

1. M. Hall, "An Isomorphism Between Linear Recurring Sequences and Algebraic Rings," *Amer. Math. Monthly*, 44 (1938), pp. 196–217.
2. N. Zierler, "Linear Recurring Sequences," *J. Soc. Indust. Appl. Math.*, 7 (1959), pp. 31–48.

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A FIBONACCI PLEASANTRY

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In the Fibonacci sequence $F_0 = 0, F_1 = 1, \cdots, F_n = F_{n-1} + F_{n-2}$, list the sums $F_n + n$ in ascending order of n and note the second differences. Do the same with $F_n - n$.

0 + 0 = 0			0 − 0 = 0		
	> 2			> 0	
1 + 1 = 2		> −1	1 − 1 = 0		> −1
	> 1			> −1	
1 + 2 = 3		> 1	1 − 2 = −1		> 1
	> 2			> 0	
2 + 3 = 5		> 0	2 − 3 = −1		> 0
	> 2			> 0	
3 + 4 = 7		> 1	3 − 4 = −1		> 1
	> 3			> 1	
5 + 5 = 10		> 1	5 − 5 = 0		> 1
	> 4			> 2	
8 + 6 = 14		> 2	8 − 6 = 2		> 2
	> 6			> 4	
13 + 7 = 20		> 3	13 − 7 = 6		> 3
	> 9			> 7	
21 + 8 = 29		> 5	21 − 8 = 13		> 5
	> 14			> 12	
34 + 9 = 43		> 8	34 − 9 = 25		> 8
	> 22			> 20	
55 + 10 = 65		> 13	55 − 10 = 45		> 13

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