ON CONTINUED FRACTION EXPANSIONS WHOSE ELEMENTS ARE ALL ONES

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I. EVEN PERIOD EXPANSIONS

1. NUMBER THEORY REVIEW. Here is an example of an even continued fraction expansion of $\sqrt{D}$, D a non-square integer, with $D = 13$.

$$\sqrt{13} = 3 + \sqrt{13} - 3 = 3 + \frac{\sqrt{13}+3}{4}$$

$$\frac{\sqrt{13}+3}{4} = 1 + \frac{\sqrt{13}-1}{4} = 1 + \frac{\sqrt{13}+1}{3}$$

$$\frac{\sqrt{13}+1}{3} = 1 + \frac{\sqrt{13}-2}{3} = 1 + \frac{\sqrt{13}+2}{3}$$

$$\frac{\sqrt{13}+2}{3} = 1 + \frac{\sqrt{13}-1}{3} = 1 + \frac{\sqrt{13}+1}{4}$$

$$\frac{\sqrt{13}+1}{4} = 1 + \frac{\sqrt{13}-3}{4} = 1 + \frac{\sqrt{13}+1}{1}$$

Hence $\sqrt{13} = \langle 3, 1, 1, 1, 1, 6 \rangle$ and the solution of the Pellian equations $x^2 - Dy^2 = d_i$ can be found from the table.

continued fraction elements c_i	3	1	1	1	1	6
signed denominators d_i	−4	3	−3	4	−1	
p convergents p_i	3	4	7	11	18	
q convergents q_i	1	1	2	3	5	

The q convergents are the Fibonacci numbers. The primitive solution of $x^2 - 13y^2 = -1$ is picked up from the half period. Thus

$$y = 1^2 + 2^2 = 5; \qquad x = 4 \times 1 + 7 \times 2 = 18.$$

In general for period $2r$,

$$y = q_r^2 + q_{r-1}^2 = q_{2r-1}; \qquad x = p_{r-1}q_{r-1} + p_r q_r = q_{2r-1}\,.$$

Also the representation of D as the sum of two squares can be found as

$$D = d_r^2 + (D - d_r^2) = d_r^2 + t^2\,,$$

where d_r is the middle denominator. Thus $13 = 3^2 + 2^2$. Finally for $D = 5$ (modulo 8), since a signed denominator is ± 4, the convergents under the -4 column are the coefficients of the cubic root of unity

$$\frac{3+\sqrt{13}}{2}$$

in the field $(1, \sqrt{13})$.

Since the period is even the x_0 of the quadratic congruence $x_0^2 \equiv -1 \pmod{13}$ is given by $x_0 \equiv x \equiv 18 \equiv 5$ (modulo 13).

2. FIBONACCI RELATIONS TO BE USED.

(a) $$(F_n, F_{n+1}) = 1.$$

(b) $$F_{2n}^2 + 1 = F_{2n-1}F_{2n+1}$$

(c) $$F_n^2 + F_{n+1}^2 = F_{2n+1}\ .$$

It may be noted that no odd Fibonacci number is ever divisible by a prime of the form $p = 4s + 3$ since from (b) $x^2 \equiv -1 \pmod p$ which is impossible.

3. EVEN VARIABLE DIFFERENCE TABLE: $D = m^2 + k$

			2n ones $n > 1$	
m	1	1 .... 1 1		$2m$
$-k$	———————————		-1	
m	$m+1$	$2m+1$	$mF_{2n+1} + F_{2n}$	
1	1	2	F_{2n+1}	

The supposition $(mF_{2n+1} + F_{2n})^2 - F_{2n+1}^2(m^2 + k) = -1$ leads to

$$2mF_{2n}F_{2n+1} + F_{2n}^2 - kF_{2n+1}^2 = -1$$

$$2mF_{2n}F_{2n+1} - kF_{2n+1}^2 = -(F_{2n}^2 + 1) = -F_{2n-1}F_{2n+1}$$

$$2mF_{2n} - kF_{2n+1} = F_{2n-1}$$

Recalling that $(F_n, F_{n+1}) = 1$ and that F_{3n} is always even this linear diophantine equation will have an infinite number of positive integer solutions for m and k unless $2n + 1 \equiv 0 \pmod 3$.

Example. $$D = m^2 + k, \qquad \sqrt{D} = \langle m, 1, 1, 1, 1, 1, 1, 2m \rangle$$

$$(13m + 8)^2 - 169(m^2 + k) = -1$$

$$16m - 13k = -5, \qquad k = m + \frac{3m+5}{13}$$

$$m = 7, \quad k = 7 + 2 = 9, \quad D = 58, \quad \sqrt{58} = \langle 7, 1, 1, 1, 1, 1, 1, 14 \rangle, \quad x^2 - 58y = -1$$

has primitive solution

$$x = 13m + 8 = 99, \quad y = 13.$$

$$m = 13 + 7 = 20, \quad k = 20 + 5 = 25, \quad D = 425, \quad \sqrt{425} = \langle 20, 1, 1, 1, 1, 1, 1, 40 \rangle, \quad x^2 - 425y^2 = -1$$

has primitive solution

$$x = 13m + 8 = 268, \quad y = 13.$$

In general if

$$D = 169m^2 - 140m + 29, \qquad \sqrt{D} = \langle 13m - 6, 1, 1, 1, 1, 1, 1, 26m - 12 \rangle$$

and the primitive solution of $x^2 - Dy^2 = -1$ is given by $x = 169m - 70$, $y = 13$.

II. ODD PERIOD EXPANSIONS

4. NUMBER THEORY REVIEW. Let $D = 135$

$$\sqrt{135} = 11 + \sqrt{135} - 11 = 11 + \frac{\sqrt{135} + 11}{14}$$

$$\frac{\sqrt{135} + 11}{14} = 1 + \frac{\sqrt{135} - 3}{14} = 1 + \frac{\sqrt{135} + 3}{9}$$

$$\frac{\sqrt{135} + 3}{9} = 1 + \frac{\sqrt{135} - 6}{9} = 1 + \frac{\sqrt{135} + 6}{11}$$

$$\frac{\sqrt{135} + 6}{11} = 1 + \frac{\sqrt{135} - 5}{11} = 1 + \frac{\sqrt{135} + 5}{10}$$

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$$\frac{\sqrt{135}+5}{10} = 1+\frac{\sqrt{135}-5}{10} = 1+\frac{\sqrt{135}+5}{11}$$

$$\frac{\sqrt{135}+5}{11} = 1+\frac{\sqrt{135}-6}{11} = 1+\frac{\sqrt{135}+6}{9}$$

$$\frac{\sqrt{135}+6}{9} = 1+\frac{\sqrt{135}-3}{9} = 1+\frac{\sqrt{135}+3}{14}$$

$$\frac{\sqrt{135}+3}{14} = 1+\frac{\sqrt{135}-11}{14} = 1+\sqrt{135}+11$$

$$\sqrt{135}+11 = 22$$

$$\sqrt{135} = <11, 1, 1, 1, 1, 1, 1, 1, 22> \quad .$$

The solutions of the Pellian equations $x^2 - Dy^2 = d_i$ can be found from the table.

c. f. elements	c_i	11	1	1	1	1	1	1	1	22
signed denominators	d_i	−14	9	−11	10	−11	9	−14	1	
p convergents	p_i	11	12	23	35	58	93	151	244	
q convergents	q_i	1	1	2	3	5	8	13	21	

The primitive solution of $x^2 - 135y^2 = 1$ is given by $x = p_8 = 244$, $y = q_8 = 21$. It can also be picked up from the half period. If the period is $2r+1$, $y = (q_r + q_{r-2})q_{r-1}$. Here

$$y = 3(2+5) = 21,$$

$$x = q_{r-1}p_{r-2} + q_r p_{r-1}.$$

Here $x = 3 \times 23 + 5 \times 35 = 244$.

5. FIBONACCI IDENTITIES TO BE USED.

(a) $$(F_{r-2}+F_r)F_{r-1} = F_{2r-2}$$

(b) $$F_{2n-1}^2 - 1 = F_{2n}F_{2n-2}$$

6. ODD VARIABLE DIFFERENCE TABLE : $D = m^2 + k$

m	1	$2r-1$ ones	1	$2m$
$-k$			1	
m	$m+1$	$2m+1$	$mF_{2r}+F_{2r-1}$	
1	1	2	F_{2r}	

The supposition $(mF_{2r}+F_{2r-1})^2 - F_{2r}^2(m+k) = 1$ leads to

$$2mF_{2r}F_{2r-1} + F_{2r-1}^2 - kF_{2r}^2 = 1$$

$$2mF_{2r}F_{2r-1} - F_{2r}^2 k = -(F_{2r-1}^2 - 1) = -F_{2r}F_{2r-2}$$

$$2mF_{2r-1} - kF_{2r} = -F_{2r-2}$$

Since $(F_{2r}, F_{2r-1}) = 1$, this linear diophantine equation will have an infinite number of positive integer solutions unless r is a multiple of 3. When $r = 3t$, F_{2r} is even, but F_{2r-2} is odd.

Example: $D = m^2 + k$, $\sqrt{D} = <m, 1, 1, 1, 2m>$ $(3m+2)^2 - 9(m^2+k) = 1$

$$4m - 3k = -1, \quad k = m + \frac{m+1}{3}$$

$$m = 2, \quad k = 3, \quad D = 7, \quad \sqrt{7} = <2, 1, 1, 1, 4>.$$

$x^2 - 7y^2 = 1$ has solution $x = 3 \times 2 + 2 = 8 \quad y = 3.$

Since $m = 2+3 = 5, \quad k = 5+2 = 7, \quad D = 32$ follows from $k = m + \frac{m+1}{3}$.

$x^2 - 32y^2 = 1$ has primitive solution $x = 3 \times 5 + 2 = 17, \quad y = 3.$ In general,

$$D = 9m^2 - 2m, \quad \sqrt{D} = \langle 3m-1, 1, 1, 1, 6m-2 \rangle .$$

The primitive solution of $x^2 - Dy^2 = 1$ is tiven by $x = 9m-1, \quad y = 3.$

7. $D = m^2 + k, \quad 2mF_r - kF_{r+1} = -F_{r-1}$

$$\sqrt{D} = m + \sqrt{D} - m = m + \frac{\sqrt{D}+m}{k}$$

$$\frac{\sqrt{D}+m}{k} = 1 + \frac{\sqrt{D}-(k-m)}{k} = 1 + \frac{\sqrt{D}+k-m}{2m+1-k}$$

$$\frac{\sqrt{D}+k-m}{2m+1-k} = 1 + \frac{\sqrt{D}-(3m+1-2k)}{2m+1-k} = 1 + \frac{\sqrt{D}+3m+1-2k}{4k-4m-1}$$

$$\frac{\sqrt{D}+3m+1-2k}{4k-4m-1} = 1 + \frac{\sqrt{D}-(6k-7m-2)}{4k-4m+1} = 1 + \frac{\sqrt{D}+6k-7m-2}{12m-9k+4}$$

$$\frac{\sqrt{D} + F_sF_{s-1}k - (1+2F_1F_2+\cdots+2F_{s-2}F_{s-1})m - (F_1^2+F_2^2+\cdots+F_{s-2}^2)}{2mF_sF_{s-1} - kF_s^2 + F_{s-1}^2}$$

(A)

$$= 1 + \frac{\sqrt{D} - [(1+2F_1F_2+\cdots+2F_{s-1}F_s\ m) - F_sF_{s+1}k + (F_1^2F_2^2+\cdots+F_{s-1})]}{2mF_sF_{s-1} - kF_s^2 + F_{s-1}^2}$$

$$= 1 + \frac{D + (\mathrm{A})}{kF_{s+1}^2 - 2mF_sF_{s+1} - F_s^2} \quad .$$

For this last assumption to be valid,

$$(2mF_sF_{s-1} - kF_s^2 + F_{s-1}^2)(kF_{s+1}^2 - 2mF_{s+1}F_s - F_s^2) \equiv m^2 + k - (\mathrm{A})^2 .$$

This identity will be proved by equating coefficients:

1. Coefficient of $-m^2$

$$4F_s^2F_{s-1}F_{s+1} = 4F_s^2[F_s^2+(-1)^s] = 4F_s^4+4(-1)^sF_s^2 = \frac{4}{25}(L_{4s}+L_{2s}-4) = [F_{s+2}F_s - F_{s+1}F_{s-2}]^2 - 1 .$$

2. Coefficient of $-k^2$

$$F_s^2\,F_{s+1}^2 = F_s^2F_{s+1}^2 .$$

3. Constant term:

$$-F_s^2F_{s-1}^2 = -(F_1^2+F_2^2+\cdots+F_{s-1}^2)^2 .$$

4. Coefficient of $2mk$

$$F_{s-1}F_sF_{s+1}^2 + F_s^3F_{s+1} = F_sF_{s+1}(F_{s-1}F_{s+1}+F_s^2) = [2L_{2s}+(-1)^s]F_sF_{s+1}$$

$$F_sF_{s+1}(1+2F_1F_2+\cdots+2F_{s-1}F_s = F_sF_{s+1}(F_{s+2}F_s - F_{s+1}F_{s-2}) = [2L_{2s}+(-1)^s].F_sF_{s+1}$$

5. Coefficient of k.

$$2F_sF_{s+1}(F_1^2+F_2^2+\cdots+F_{s-1}^2)+1 = 2F_s^2F_{s-1}F_{s+1}+1 = 1+2F_s^2[F_s^2+(-1)^s] = 2F_s^4+2F_s^2(-1)^s+1$$

$$F_{s-1}^2F_{s+1}^2 + F_s^4 = F_s^4 + [F_s^2+(-1)^s]^2 = 2F_s^4 + 2(-1)^sF_s^2 + 1 .$$

6. Coefficient of $-2m$

$$F_s^3F_{s-1} + F_{s-1}^2F_sF_{s+1} = F_{s-1}F_s[F_s^2 + F_{s-1}F_{s+1}] = F_{s-1}F_s[F_s(F_{s+2} - F_{s+1}) + F_{s-1}F_{s+1}]$$

$$= F_{s-1}F_s[F_sF_{s+2} - F_{s+1}(F_s - F_{s-1})] = F_{s-1}F_s(F_sF_{s+2} - F_{s+1}F_{s-2}).$$

$$(F_1^2 + F_2^2 + \cdots + F_{s-1}^2)(1 + 2F_1F_2 + 2F_sF_3 + \cdots + 2F_{s-1}F_s = F_{s-1}F_s[F_sF_{s+2} - F_{s+1}F_{s-2}]$$

In proving this identity the following Fibonacci identities were used:

(a) $$1 + 2F_1F_2 + \cdots + 2F_{s-1}F_s = F_sF_{s+2} - F_{s+1}F_{s-2}$$

(b) $$F_1^2 + F_2^2 + \cdots + F_s^2 = F_{s-1}F_s$$

(c) $$F_{s-1}F_{s+1} = F_s^2 + (-1)^s \;.$$

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A MORE GENERAL FIBONACCI MULTIGRADE

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In a recent article I gave examples of multigrades based on Fibonacci series in which

$$F_{n+2} = F_{n+1} + F_n \;.$$

Here I first give a more general multigrade for series in which

$$F_{n+2} = yF_{n+1} + xF_n \,.$$

Consider

1 3 7 17 47 (where $x = 1, y = 2$).

By inspection we notice that

$$1^m + 3^m + 3^m + 7^m = 0^m + 4^m + 4^m + 6^m$$

$$3^m + 7^m + 7^m + 17^m = 0^m + 10^m + 10^m + 14^m \text{, etc.}$$

(where $m = 1, 2$).

We can look at other series of a like kind:

1 3 10 33 109 (where $x = 1,\ y = 3$).

Here

$$1^m + 3^m + 3^m + 3^m + 10^m + 10^m = 0^m + 0^m + 7^m + 7^m + 7^m + 9^m$$

$$3^m + 10^m + 10^m + 10^m + 33^m + 33^m = 0^m + 0^m + 23^m + 23^m + 23^m + 30^m, \text{ etc.}$$

(where $m = 1, 2$)

1 3 11 39 139 (where $x = 2,\ y = 3$).

Here

$$1^m + 1^m + 3^m + 3^m + 3^m + 11^m + 11^m + 11^m = 0^m + 0^m + 0^m + 8^m + 8^m + 8^m + 10^m + 10^m$$

$$3^m + 3^m + 11^m + 11^m + 11^m + 39^m + 39^m + 39^m = 0^m + 0^m + 0^m + 28^m + 28^m + 28^m + 36^m + 36^m, \text{ etc.}$$

(where $m = 1, 2$)

The general series

$$a \qquad b \qquad ax + by \qquad bx + axy + by^2$$

gives

$$x(a)^m + y(b)^m + (x + y - 2)(ax + by)^m = (x + y - 2)0^m + y(ax + by - b)^m + x(ax + by - a)^m$$

(where $m = 1, 2$).

Continued on page 66.