TREBLY-MAGIC SYSTEMS IN A LATIN 3-CUBE OF ORDER EIGHT

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The Tarry-Escott problem requires that for each positive integer t the least integer $N(t)$ be found such that there exist two distinct sets of integers $\{a_i\}, \{b_i\}, i = 1 \cdots N(t)$ such that $a_i^m = b_i^m$ for $m = 1 \cdots t$. It is easily shown that for each t, $N(t) \geq t + 1$ and that for small values of t equality holds. For example $N(2) = 3$ since the sets $\{1,8,9\}$ and $\{3,4,11\}$ satisfy the equations $1 + 8 + 9 = 3 + 4 + 11$ and $1^2 + 8^2 + 9^2 = 3^2 + 4^2 + 11^2$. A complete solution to the problem is unknown.

We call a system $L = \{S_i\}_{i=1}^n$ of sets of integers t-magic if the numbers

$$\sum_{s \in S_t} s^m$$

are independent of the choice of S_i for $m = 1 \cdots t$. Thus a solution to the Tarry-Escott problem is a t-magic system of two sets of cardinality $N(t)$.

It has been shown [1] that for appropriate choices of n and k, orthogonal systems of magic Latin k-cubes of order n can be constructed. In this paper we exhibit a Latin 3-cube of order 8 in which are embedded subcubes possessing hypermagic properties.

The cube (Fig. 1) comprises 8^3 ordered triples with entries 0,1,2,3,4,5,6,7. It is orthogonal, viz., each of the triples from 000 to 777 appears exactly once. In the diagram we show the cube as a set of eight squares which are to be placed one above the other to form the complete 3-dimensional array. After each of the entries is attached one of the letters *a, b, c, d.* Each of the rows in each square is labeled with one of the symbols $R_{00}, R_{01}, R_{11}, R_{20}, R_{21}, R_{30}, R_{31}$ and each of the columns is labeled with one of $K_{00}, K_{01}, \cdots, K_{31}$. Thus the totality of entries R_{ij} represents a set of rows parallel to one of the horizontal edges of the cube. A similar statement can be made about all entries labeled K_{ij}.

The two subcubes that we consider are designated as A and B. They are constructed as follows. Cube A is obtained by deleting the second entry in each cell of the original cube and regarding the remaining pair as a two-digit number in base eight. So that each of the first 64 positive integers may appear in each subsquare of the cube we add 1 to each of the two-digit numbers. Thus the first row of the first square of cube A is: *20a 33b 76c 51d 44a 67b 22c 05d* R_{00}. Cube B is constructed exactly the same way, deleting the first entry in each cell. For convenience in computation we convert the entries to base ten.

We denote by A_k the k^{th} (horizontal) square of cube A and by B_k the k^{th} square of cube B. Then a_{ijk} is the entry in the i^{th} row, k^{th} column of A_k and b_{ijk} the corresponding entry in B_k.

We now observe that for fixed j,k

$$\sum_i a_{ijk} = \sum_i b_{ijk} = 260$$

	K_{00}	K_{01}	K_{10}	K_{11}	K_{20}	K_{21}	K_{30}	K_{31}	
1	117a	332b	775c	550d	443a	666b	221c	004d	R_{00}
2	532a	717b	350c	175d	066a	243b	604c	421d	R_{10}
3	050b	275a	632d	417c	504b	721a	366d	143c	R_{11}
4	475b	650a	217d	032c	121b	304a	743d	566c	R_{30}
5	643c	466d	021a	204b	317c	132d	575a	750d	R_{20}
6	266c	043d	404a	621b	732c	517d	150a	375b	R_{01}
7	704d	521c	166b	343a	250d	075c	432b	617a	R_{31}
8	321d	104c	543b	766a	675d	450c	017b	232a	R_{21}

Square 1

	K_{00}	K_{01}	K_{10}	K_{11}	K_{20}	K_{21}	K_{30}	K_{31}	
1	425b	600a	247d	062c	171b	354a	713d	536c	R_{30}
2	000b	225a	662d	447c	554b	771a	336d	113c	R_{10}
3	562a	747b	300c	125d	036a	213b	654c	471d	R_{11}
4	147a	362b	725c	500d	413a	636b	271c	054d	R_{00}
5	371d	154c	513b	736a	625d	400c	047b	262a	R_{20}
6	754d	571c	136b	313a	200d	025c	462b	647a	R_{31}
7	236c	031d	454a	671b	762c	547d	100a	325b	R_{01}
8	613c	436d	071a	254b	347c	162d	525a	700b	R_{21}

Square 2

1	361d	144c	503b	726a	635d	410c	057b	272a	R_{20}
2	744d	561c	126b	303a	210d	035c	472b	657a	R_{30}
3	226c	003d	444a	661b	772c	557d	110a	335b	R_{00}
4	603c	426d	061a	244b	357c	172d	535a	710b	R_{21}
5	435b	610a	257d	072c	161b	344a	703d	526c	R_{31}
6	010b	235a	672d	457c	544b	761a	326d	103c	R_{10}
7	572a	757b	310c	135d	026a	203b	644c	461d	R_{11}
8	157a	372b	735c	510d	403a	626b	261c	044d	R_{01}

Square 3

1	653c	476d	031a	214b	307c	122d	565a	740b	R_{20}
2	276c	053d	414a	631b	722c	507d	140a	365b	R_{00}
3	714d	531c	176b	353a	240d	065c	422b	670a	R_{30}
4	331d	114c	553b	776a	665d	440c	007b	222a	R_{21}
5	107a	322b	765c	540d	453a	676b	231c	014d	R_{01}
6	522a	707b	340c	165d	076a	253b	614c	431d	R_{10}
7	040b	265a	622d	407c	514b	731a	376d	153c	R_{11}
8	465b	640a	207d	022c	131b	314a	753d	576c	R_{31}

Square 4

1	206c	023d	464a	641b	752c	577d	130a	315b	R_{00}
2	623c	406d	041a	264b	377c	152d	515a	730b	R_{20}
3	341d	164c	523b	706a	615d	430c	077b	252a	R_{21}
4	764a	541c	106b	323a	230d	015c	452b	677a	R_{30}
5	522a	777b	330c	115d	006a	223b	664c	441d	R_{10}
6	177a	352b	715c	530d	423a	606b	241c	064d	R_{01}
7	415b	630a	277d	052c	141b	364a	723d	506c	R_{31}
8	030b	215a	652d	477c	564b	741a	306d	123c	R_{11}

Square 5

1	734d	511c	156b	373a	260d	045c	402b	627a	R_{30}
2	311d	134c	573b	756a	645d	460c	027b	202a	R_{20}
3	673c	456d	011a	234b	327c	102d	545a	760b	R_{21}
4	256c	073d	434a	611b	702c	527d	160a	345b	R_{00}
5	060b	245a	602d	427c	534b	711a	356d	173c	R_{10}
6	445b	660a	227d	002c	111b	334a	773d	556c	R_{31}
7	127a	302b	745c	560d	473a	656b	211c	034d	R_{01}
8	502a	727b	360c	145d	056a	273b	634c	411d	R_{11}

Square 6

1	070b	255a	612d	437c	524b	701a	346d	163c	R_{10}
2	455b	670a	237d	012c	101b	324a	763d	546c	R_{30}
3	137a	312b	755c	570d	463a	646b	201c	024d	R_{00}
4	512a	737b	370c	155d	046a	263b	624c	401d	R_{11}
5	724d	501c	146b	363a	270d	055c	412b	637a	R_{31}
6	301d	124c	563b	746a	655d	470c	037b	212a	R_{20}
7	663c	446d	001a	224b	337c	112d	555a	770b	R_{21}
8	246c	063d	424a	601b	712c	537d	170a	355b	R_{01}

Square 7

1	542a	767b	320c	105d	016a	233b	674c	451d	R_{10}
2	167a	342b	705c	520d	433a	616b	251c	074d	R_{00}
3	405b	620a	267d	042c	151b	374a	733d	516c	R_{30}
4	020b	205a	642d	467c	574b	751a	316d	133c	R_{11}
5	216c	033d	474a	651b	742c	537d	120a	305b	R_{01}
6	633c	416d	051a	274b	367c	142d	505a	720b	R_{20}
7	351d	174c	533b	716a	605d	420c	067b	242a	R_{21}
8	774d	551c	116b	333a	220d	005c	442b	667a	R_{31}

Square 8

Figure 1

and for fixed i,k

$$\sum_j a_{ijk} = \sum_j b_{ijk} = 260.$$

Similarly

$$\sum_i a_{ijk}^2 = \sum_i b_{ijk}^2 = \sum_j a_{ijk}^2 = \sum_j b_{ijk}^2 = 11180.$$

Thus in a natural way, we have exhibited a system of 256 sets of eight integers that is 2-magic.

We now define a system of 196 sets of 16 integers that is 3-magic. This system has the pleasant property that it includes the principal diagonals as well as the rows and columns of cubes A and B.

Let A_{ka} be the set of 16 numbers in A_k that are followed by the letter a. Let $A_{kb}, A_{kc}, A_{kd}, B_{ka}, B_{kb}, B_{kc}, B_{kd}$ be similarly defined. (This defines 64 sets.)

Let AR_{ki} (resp. B_{ki}) be the set of 16 numbers in A_k (resp. B_k) that lie in rows R_{i0} or R_{i1}, $i = 0, 1, 2, 3$. (This defines 64 sets.) Let AK_{ki} (resp. B_{ki}) be the set of 16 numbers in A_k (resp. B_k) that lie in columns K_{i0} or K_{i1}, $i = 0, 1, 2, 3$. (This defines 64 sets.)

Let AD_a (resp. BD_a) be the set of numbers in the two main diagonals of cube A (resp. B) of the form a_{iii} or $a_{8-i,8-i,i}$ (resp. b_{iii}, $b_{8-i,8-i,i}$). It will be observed that each of these entries is labeled by the letter a. Similarly let AD_d (resp. BD_d) be the set in the other two main diagonals

$$\{a_{i,8-i,i}\},\quad \{a_{8-i,i,i}\},\quad \{b_{i,8-i,i}\},\quad \{b_{8-i,i,i}\}.$$

(This defines 4 sets.)

Now let L be the system of 196 sets defined above. It can be verified that L is a 3-magic system. Explicitly, if $S \in L$ then

$$\sum_{s\in S} s = 520,\qquad \sum_{s\in S} s^2 = 22360 \quad\text{and}\quad \sum_{s\in S} s^3 = 1081600.$$

We remark in conclusion that we have by no means exhausted the hypermagic systems that can be extracted from the cubes. To this end we append the following constructions.

HYPERMAGIC CONSTRUCTIONS

In what follows, when it is mentioned that sets of numbers (in this case each set contains 16 two-digit numbers) are equal in sum, this will mean that they have the same sum of k^{th} powers for $k = 1, 2$ and 3.

We also point out that each row in every one of the eight squares has two numbers that end in a, two numbers that end in b, two numbers that end in c, and two numbers that end in d.

I	1	2	3	4	5	6	7	8
II	2	1	4	3	6	5	8	7
III	2	7	4	5	6	3	8	1
IV	3	4	1	2	7	8	5	6
V	4	5	2	7	8	1	6	3
VI	4	3	2	1	8	7	6	5
VII	5	6	7	8	1	2	3	4
VIII	6	3	8	1	2	7	4	5
IX	6	5	8	7	2	1	4	3
X	7	8	5	6	3	4	1	2
XI	8	7	6	5	4	3	2	1
XII	8	1	6	3	4	5	2	7
Z	1	2	3	4	5	6	7	8 = Square Number

Figure 2 Chart

How to Read the Figure 2 Chart

The numbers on the bottom of the Chart (below line *Z*) each denotes the number of some square in the cube. The number in the column above the number denoting a square denotes a row number (counting from top to bottom) in the particular square listed on the bottom of the column. For example: Cell (VII,6) = 2 denotes the 8 numbers on row 2 to Square 6. *Each of the 6 numbers on a row in the Chart represents a magic system.* For example: We write the numbers on row VII to get row 5 in square 1, row 6 in square 2, ··· row 4 in square 8. We now arrange the (resulting) 64 3-digit numbers so that the 16 numbers that end in *a* are in (say) column 1, the 16 numbers that end in *b* are in column 2, and the 16 numbers that end in *c* are in column 3, and the 16 numbers that end in *d* are (say) in column 4.

We first consider the first and third digit of each and every number in the 4 columns (that is cube A) and after adding 1 to each pair of digits we express the 64 2-digit numbers in the scale of 10.

We now add (in cube A) the 16 numbers in column 1 to get the sum *s*1,
" " " " " " " " " " 2 " " " " *s*2,
" " " " " " " " " " 3 " " " " *s*3,
" " " " " " " " " " 4 " " " " *s*4.

Then for the sum of the k^{th} powers (for k = 1, 2 and 3) we have $s1 \overset{3}{=} s2 \overset{3}{=} s3 \overset{3}{=} s4$ (in cube A).

The exact relationship between the numbers in cube A also holds true for cube B (in the 2nd and 3rd digits).

REFERENCES

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