# RECURSION RELATIONS OF PRODUCTS OF LINEAR RECURSION SEQUENCES

**BROTHER ALFRED BROUSSEAU**
**St. Mary's College, Moraga, California 94575**

Given two sequences $S_i$ and $T_i$ governed respectively by linear recursion relations

(1) $$S_n = \sum_{i=1}^{p} a_i S_{n-i}$$

of order $p$ and

(2) $$T_n = \sum_{i=1}^{q} b_i T_{n-1}$$

of order $q$. Required to find the recursion relation of the term-by-term product of the two sequences $Z_i = S_i T_i$.

Initially we shall assume that the roots of the auxiliary equations corresponding to the above recursion relations are distinct so that:

(3) $$S_n = \sum_{i=1}^{p} A_i s_i^n ,$$

where $s_i$ $(i = 1,p)$ are the roots of $x^p - a_1x^{p-1} - a_2x^{p-2} \cdots - a_p = 0$. Similarly,

(4) $$T_n = \sum_{i=1}^{q} B_i t_i^n ,$$

where $t_i$ $(i = 1,q)$ are the roots of $x^q - b_1x^{q-1} - b_2x^{q-2} \cdots - b^q = 0$.

No universal formulation applying to all orders has been arrived at so that the results will be given as a series of algorithms applying to particular cases. The method employed is to find the products of the general terms (3) and (4) and then note the new set of roots for the recursion relation of the product. By finding the symmetric functions of these roots one can arrive at the recursion relation of the term-by-term product.

## 1. GEOMETRIC PROGRESSION BY ANOTHER SEQUENCE

A geometric progression is a linear recursion relation of the first order:

$$S_n = rS_{n-1}$$

whose general term can be taken as $S_n = Ar^n$. If such a term be multiplied by (4) one has:

(5) $$S_n T_n = \sum_{i=1}^{q} B_i (rt_i)^n$$

Thus these terms behave as belonging to an auxiliary equation whose roots are $rt_i$ $(i = 1,q)$. Consequently by finding the symmetric functions of these quantities one arrives at the linear recursion relation governing the terms $Z_n = S_n T_n$. It is not difficult to verify that this leads to:

(6) $$Z_n = \sum_{i=1}^{q} B_i r^i Z_{n-i}$$

## 2. TWO RELATIONS OF THE SECOND ORDER

Let the auxiliary equations corresponding to two linear recursion relations of the second order be:

$$x^2 + a_1 x + b_1 = 0$$
$$x^2 + a_2 x + b_2 = 0$$

Let the terms of the sequence governed by the first relation be:

$$S_n = Ar^n + Bs^n$$

and the terms governed by the second sequence be:

$$T_n = Cu^n + Dv^n .$$

Then

$$Z_n = S_n T_n = AC(ru)^n + AD(rv)^n + BC(su)^n + BD(sv)^n .$$

The roots of the auxiliary equation for $Z_n$ are $ru, rv, su, sv$. To obtain the coefficients of this equation we calculate the symmetric functions of these roots.

$$S_{4,1} = (r+s)(u+v) = (-a_1)(-a_2) = a_1 a_2$$

$$S_{4,2} = r^2uv + rsu^2 + rsuv + rsuv + rsv^2 + s^2uv = uv(r^2+s^2) + rs(u^2+v^2) + 2rsuv$$
$$= b_1(a_2^2 - 2b_2^2) + b_2(a_1^2 - 2b_1) + 2b_1b_2 = b_1a_2^2 + b_2a_1^2 - 2b_1b_2$$

$$S_{4,3} = r^2su^2v + r^2suv^2 + rs^2u^2v + rs^2uv^2 = rsuv(r+s)(u+v) = b_1b_2a_1a_2$$

$$S_{4,4} = r^2s^2u^2v^2 = b_1^2b_2^2 .$$

The recursion relation for the product of two sequences of the second order is thus

$$x^4 - a_1a_2x^3 + (b_1a_2^2 + b_2a_1^2 - 2b_1b_2)x^2 - a_1a_2b_1b_2x + b_1^2b_2^2 = 0 .$$

EXAMPLE. The sequence 1, 4, 17, 72, 305, ⋯ is governed by $T_{n+1} = 4T_n + T_{n-1}$ while 1, −5, 26, −135, 701, ⋯ is governed by $T_{n+1} = -5T_n + T_{n-1}$. The product sequence is 1, −20, 442, −9720, 213805, ⋯. In terms of the above formulation, $a_1 = -4$, $b_1 = -1$, $a_2 = 5$, $b_2 = -1$. The auxiliary equation for the product sequence is given by:

$$x^4 + 20x^3 - 43x^2 + 20x + 1 = 0.$$

$$(-9720)(-20) + 442*43 + (-20)(-20) - 1 = 213805 .$$

## SECOND- AND THIRD-ORDER RECURSION RELATIONS

Given two sequences $S_n$, $T_n$ governed respectively by the relations

(7) $$x^2 + a_1x + b_1 = 0$$

(8) $$x^3 + a_2x^2 + b_2x + c_2 = 0$$

with roots $r_1, s_1$ and $r_2, s_2, t_2$, respectively. The recursion relation of the product $S_nT_n$ will have for roots $r_1r_2$, $r_1s_2$, $r_1t_2$, $s_1r_2$, $s_1s_2$, $s_1t_2$. The symmetric functions of these roots are as follows.

(9) $$S_{6,1} = (r_1+s_1)(r_2+s_2+t_2) = a_1a_2$$

(10) $$S_{6,2} = r_1s_1(s_2^2 + t_2^2 + r_2^2) + (r_1^2 + s_1^2)(r_2s_2 + r_2t_2 + s_2t_2) + 2r_1s_1(r_2s_2 + r_2t_2 + s_2t_2)$$
$$= b_1(a_2^2 - 2b_2) + (a_1^2 - 2b_1)b_2 + 2b_1b_2 = b_1a_2^2 + b_2a_1^2 - 2b_1b_2$$

(11) $$S_{6,3} = (r_1^3 + s_1^3)(r_2s_2t_2) + r_1s_1(r_1+s_1)(r_2+s_2+t_2)(r_2s_2+r_2t_2+s_2t_2)$$
$$= (a_1^3 - 3a_1b_1)c_2 + a_1b_1a_2b_2$$

(12) $$S_{6,4} = r_1s_1(r_1^2+s_1^2)r_2s_2t_2 \quad +s_2+t_2)+r_1^2s_1^2(r_2s_2+r_2t_2+s_2t_2)^2$$
$$= b_1a_1^2a_2c_2+b_1^2b_2^2-2b_1^2a_2c_2$$

(13) $$S_{6,5} = r_1^2s_1^2(r_1+s_1)r_2s_2t_2(r_2s_2+r_2t_2+s_2t_2) = b_1^2a_1b_2c_2$$

(14) $$S_{6,6} = r_1^3s_1^3r_2^2s_2^2t_2^2 = b_1^3c_2^2$$

EXAMPLE

| $x^2-4x+3=0$ | $x^3-3x^2+6x-3=0$ | |
|---|---|---|
| 2 | 3 | 6 |
| 7 | –21 | -147 |
| 22 | –60 | -1320 |
| 67 | –45 | -3015 |
| 202 | 162 | 32724 |
| 607 | 576 | 349632 |
| 1822 | 621 | 1131462 |
| 5467 | –1107 | –6051969 |
| 16402 | –5319 | –87242238 |

$$a_1=-4, \qquad b_1=3, \qquad a_2=-3, \qquad b_2=6, \qquad c_2=-3$$

$$S_{6,1} = 12$$
$$S_{6,2} = 3*9+6*16-2*18 = 87$$
$$S_{6,3} = (-64+36)(-3)+216 = 300$$
$$S_{6,4} = 3*16*9+9*36-2*9*9 = 594$$
$$S_{6,5} = 9(-4)*6(-3) = 648$$
$$S_{6,6} = 27*9 = 243\,.$$

The recursion relation corresponds to:

$$x^6-x^5-12x^5+87x^4-300x^3+594x^2-648x+243 = 0.$$

CHECK

$$12(-6051969)-87(1131462)+300(349632)-594(32724)+648(-3015)-243(-1320) = -87242238\,.$$

## TWO THIRD-ORDER RELATIONS

For two sequences governed by the relations:

$$x^3+a_1x^2+b_1x+c_1 = 0 \qquad \text{and} \qquad x^3+a_2x^2+b_2x+c_2 = 0$$

The coefficients of the recursion relation of the product are found to be:

| | |
|---|---|
| $x^9$ | $1$ |
| $x^8$ | $-a_1a_2$ |
| $x^7$ | $a_1^2b_2+a_2^2b_1-2b_1b_2$ |
| $x^6$ | $-a_1^3c_2-a_2^3c_1-3c_1c_2+3a_1b_1c_2-3a_2b_2c_1-a_1a_2b_1b_2$ |
| $x^5$ | $b_1^2b_2^2-2a_2b_1^2c_2-2a_1b_2^2c_1+a_1^2a_2b_1c_2+a_1a_2^2b_2c_1-a_1a_2c_1c_2$ |
| $x^4$ | $-a_1b_1^2b_2c_2-a_2b_1b_2^2c_1+2a_1^2\;{}_2c_1c_2+2a_2^2b_1c_1c_2-a_1^2a_2^2c_1c_2+b_1b_2c_1c_2$ |
| $x^3$ | $b_1^3c_2^2+b_2^3c_1^2-3a_1b_1c_1c_2^2-3a_2b_2c_1^2c_2+3c_1^2c_2^2+a_1a_2b_1b_2c_1c_2$ |
| $x^2$ | $-a_2b_1^2c_1c_2^2-a_1b_2^2c_1^2c_2+2a_1a_2c_1^2c_2^2$ |
| $x$ | $b_1b_2c_1^2c_2^2$ |
| $1$ | $-c_1^3c_2^3$ |

EXAMPLE $x^3 - 3x^2 + 5x - 2 = 0 \quad x^3 + 4x^2 - 7x - 3 = 0$

| | | |
|---|---|---|
| 1 | 0 | 0 |
| −2 | 1 | −2 |
| −9 | 2 | −18 |
| −15 | −1 | 15 |
| −4 | 21 | −84 |
| 45 | −85 | −3825 |
| 125 | 484 | 60500 |
| 142 | −2468 | −350456 |
| −109 | 13005 | −1417545 |
| −787 | −67844 | 53393228 |
| −1532 | 355007 | −543870724 |
| −879 | −1855921 | 1631354559 |
| 3449 | 9705201 | 33473238249 |

$$a_1 = -3, \quad b_1 = 5, \quad c_1 = -2, \quad a_2 = 4, \quad b_2 = -7, \quad c_2 = -3.$$

The recursion relation for the product is:

$$x^9 + 12x^8 + 87x^7 - 88x^6 + 97x^5 + 2665x^4 + 563x^3 - 828x^2 - 1260x - 216 = 0.$$

CHECK

$$-12*1631354559 - 87(-543870724) + 88*53393228 - 97(-1417545) - 2665(-350456)$$
$$- 563*60500 + 828(-3825) + 1260(-84) + 216*15 = 33473238249.$$

## SECOND AND FOURTH ORDERS

Given two sequences governed by the following relations, respectively:

$$x^2 + a_1x + b_1 = 0$$

$$x^4 + a_2x^3 + b_2x^2 + c_2x + d_2 = 0.$$

The coefficients of the product recursion relation are:

| | |
|---|---|
| $x^8$ | $1$ |
| $x^7$ | $-a_1a_2$ |
| $x^6$ | $a_1^2b_2 + a_2^2b_1 - 2b_1b_2$ |
| $x^5$ | $-a_1^3c_2 - a_1a_2b_1b_2 + 3a_1b_1c_2$ |
| $x^4$ | $a_1^4d_2 - 4a_1^2b_1d_2 + 2b_1^2d_2 + a_1^2a_2b_1c_2 - 2a_2b_1^2c_2 + b_1^2b_2^2$ |
| $x^3$ | $-a_1^3a_2b_1d_2 - a_1b_1^2b_2c_2 + 3a_1a_2b_1^2d_2$ |
| $x^2$ | $a_1^2b_1^2b_2d_2 + b_1^3c_2^2 - 2b_1^3b_2d_2$ |
| $x$ | $-a_1b_1^3c_2d_2$ |
| $1$ | $b_1^4d_2^2$ |

EXAMPLE $x^2 - 3x + 2 = 0 \quad x^4 + 2x^3 - 3x^2 + x - 3 = 0$

| | | |
|---|---|---|
| 10 | −8 | −80 |
| 22 | 34 | 748 |
| 46 | −97 | −4462 |
| 94 | 319 | 29986 |
| 190 | −987 | −187530 |
| 382 | 3130 | 1195660 |
| 766 | −9831 | −7530546 |
| 1534 | 30996 | 47547864 |
| 3070 | −97576 | −299558320 |
| 6142 | 307361 | 1887811262 |
| 12286 | −967939 | −11892098554 |

CHECK

The recursion relation of the product corresponds to:

$$x^8 + 6x^7 - 7x^6 - 27x^5 + 5x^4 - 144x^3 + 188x^2 - 72x + 144 = 0$$

$$-6(1887811262) + 7(-299558320) + 27*47547864 + (-5)(-7530546) + 144*1195660$$
$$- 188(-187530) + 72*29986 - 144(-4462) = -11892098554 .$$

## THIRD- AND FOURTH-ORDER SEQUENCES

For two sequences governed respectively by the relations corresponding to:

$$x^3 + a_1x^2 + b_1x + c_1 = 0$$

and

$$x^4 + a_2x^3 + b_2x^2 + c_2x + d_2 = 0$$

the coefficients for the auxiliary equation of the product are given by

| | |
|---|---|
| $x^{12}$ | $1$ |
| $x^{11}$ | $-a_1a_2$ |
| $x^{10}$ | $a_1^2b_2 + a_2^2b_1 - 2b_1b_2$ |
| $x^9$ | $-a_1^3c_2 - 3c_1c_2 + 3a_1b_1c_2 + 3a_2b_2c_1 - a_1a_2b_1b_2 - a_2^3c_1$ |
| $x^8$ | $a_1^4d_2 - 4a_1^2b_1d_2 - a_1a_3c_1c_2 + a_1^2a_2b_1c_2 - 2a_2b_1^2c_2 + b_1^2b_2^2 - 2a_1b_2^2c_1 + a_1a_2^2b_2c_1 + 2b_1^2d_2 + 4a_1c_1d_2$ |
| $x^7$ | $-a_1^3a_2b_1d_2 - 5a_2b_1c_1d_2 + 3a_1a_2b_1^2d_2 + a_1^2a_2c_1d_2 - a_1b_1^2b_2c_2 + 2a_1^2b_2c_1c_2 + b_1b_2c_1c_2 - a_1^2a_2^2c_1c_2 + 2a_2^2b_1c_1c_2 - a_2b_1b_2^2c_1$ |
| $x^6$ | $a_1^2b_1^2b_2d_2 - 2a_1^3b_2c_1d_2 - 2b_1^3b_2d_2 + 4a_1b_1b_2c_1d_2 - 3b_2c_1^2d_2 + a_1^3a_2^2c_1d_2 + 3a_2^2c_1^2d_2 - 3a_1a_2^2b_1c_1d_2 + b_1^3c_2^2 - 3a_1b_1c_1c_2^2 + 3c_1^2c_2^2 + b_2^3c_1^2 - 3a_2b_2c_1^2c_2 + a_1a_2b_1b_2c_1c_2$ |
| $x^5$ | $-a_1b_1^3c_2d_2 + 3a_1^2b_1c_1c_2d_2 + b_1^2c_1c_2d_2 - 5a_1c_1^2c_2d_2 + a_1a_2b_2c_1^2d_2 - a_1^2a_2b_1b_2c_1d_2 + 2a_2b_1^2b_2c_1d_2 - a_2b_1^2c_1c_2^2 + 2a_1a_2c_1^2c_2^2 - a_1b_2^2c_1^2c_2$ |
| $x^4$ | $b_1^4d_2^2 - 4a_1b_1^2c_1d_2^2 + 2a_1^2c_1^2d_2^2 + 4b_1c_1^2d_2^2 + a_1a_2b_1^2c_1c_2d_2 - 2a_1^2a_2c_1^2c_2d_2 - a_2b_1c_1^2c_2d_2 + a_1^2b_2^2c_1^2d_2 - 2b_1b_2^2c_1^2d_2 + b_1b_2c_1^2c_2^2$ |
| $x^3$ | $-a_2b_1^3c_1d_2^2 + 3a_1a_2b_1c_1^2d_2^2 - 3a_2c_1^3d_2^2 + 3b_2c_1^3c_2d_2 - a_1b_1b_2c_1^2c_2d_2 - c_1^3c_2^3$ |
| $x^2$ | $b_1^2b_2c_1^2d_2^2 - 2a_1b_2c_1^3d_2^2 + a_1c_1^3c_2^2d_2$ |
| $x$ | $-b_1c_2c_1^3d_2^2$ |
| $1$ | $c_1^4d_2^3$ |

EXAMPLE

| $x^3 - x^2 - x - 1 = 0$ | $x^4 - x^3 - x^2 - x - 1 = 0$ | PRODUCT |
|---|---|---|
| 6 | 21 | 126 |
| 11 | 39 | 429 |
| 20 | 76 | 1520 |
| 37 | 147 | 5439 |
| 68 | 283 | 19244 |
| 125 | 545 | 68125 |
| 230 | 1051 | 241730 |
| 423 | 2026 | 856998 |
| 778 | 3905 | 3038090 |
| 1431 | 7527 | 10771137 |
| 2632 | 14509 | 38187688 |
| 4841 | 27967 | 135388247 |
| 8904 | 53908 | 479996832 |
| 16377 | 103911 | 1701750447 |
| 30122 | 200295 | 6033285990 |

The recursion relation for the product corresponds to the equation:

$$x^{12} - x^{11} - 4x^{10} - 12x^9 - 17x^8 - 12x^7 - 5x^6 + 10x^5 - 7x^4 - 2x^3 + 0x^2 + x - 1 = 0.$$

CHECK

$$479996832 + 4*135388247 + 12*38187688 + 17*10771137 + 12*3038090 + 5*856998$$
$$- 10*241730 + 7*68125 + 2*19244 - 1520 + 429 = 1701750447 .$$

## REPEATED ROOTS

The case of d roots can be handled in the same way but with some modifications in the procedure for finding the symr ıctions. This discussion will be limited to the important case in which one of the sequences has a general teı... yıvcıı ıy a polynomial function. The recursion relation for such a polynomial function of the $n^{th}$ degree has its coefficients determined by the expansion of

$$(x-1)^{n+1} = 0 .$$

In other words there are $n + 1$ roots all equal to unity.

## QUADRATIC POLYNOMIAL SEQUENCE

The general procedure can be illustrated by the case of a sequence whose terms are given by a quadratic polynomial function. To keep the resulting formulas reasonably simple, let the other sequence be limited to order five and be in the form:

$$x^5 - a_1x^4 + a_2x^3 - a_3x^2 + a_4x - a_5 = 0$$

so that the quantities $a_i$ are the symmetric functions of the roots. If the roots are given by $r_i$, the general term of the sequence would be:

$$T_n = \sum A_i r_i^n .$$

If the polynomial function is $f(n) = B_1n^2 + B_2n + B_3$, the product of the terms of the two sequences is:

$$Z_n = T_n f(n) = B_1 n^2 \sum A_i r_i^n + B_2 n \sum A_i r_i^n + B_3 \sum A_i r_i^n .$$

This shows that the equation for the product has the roots $r_i$ taken three times. The problem then is to find the symmetric functions for three such sets of roots taken together. Suppose we wish to find $S_{15,9}$, the symmetric function of these fifteen roots taken nine at a time. The various cases can be found by taking the partitions of 9 into three or less parts, the largest being five (since this limitation was set on the order of the recursion relation). These partitions would be: 54, 531, 522, 441, 432, 333. Hence

$$S_{15,9} = 6a_5a_4 + 6a_5a_3a_1 + 3a_5a_2^2 + 3a_4^2a_1 + 6a_4a_3a_2 + a_3^3$$

the coefficients being determined by the multinomial coefficient corresponding to the number of ways the various groups of roots can be selected.

For the quadratic polynomial function and linear recursion relations up to the fifth order the coefficients of the product recursion relation are as follows:

$$\begin{array}{l}
1 \\
-3a_1 \\
3a_2 + 3a_1^2 \\
-[3a_3 + 6a_1a_2 + a_1^3] \\
3a_4 + 6a_1a_3 + 3a_2^2 + 3a_2a_1^2 \\
-[3a_5 + 6a_4a_1 + 6a_3a_2 + 3a_3a_1^2 + 3a_2^2a_1] \\
6a_5a_1 + 6a_4a_2 + 3a_3^2 + 3a_4a_1^2 + 6a_3a_2a_1 + a_2^3 \\
-[6a_5a_2 + 6a_4a_3 + 3a_5a_1^2 + 6a_4a_2a_1 + 3a_3^2a_1 + 3a_3a_2^2] \\
6a_5a_3 + 3a_4^2 + 6a_5a_2a_1 + 6a_4a_3a_1 + 3a_4a_2^2 + 3a_3^2a_2 \\
-[6a_5a_4 + 6a_5a_3a_1 + 3a_5a_2^2 + 3a_4^2a_1 + 6a_4a_3a_2 + a_3^3] \\
3a_5^2 + 6a_5a_4a_1 + 6a_5a_3a_2 + 3a_4^2a_2 + 3a_4a_3^2 \\
-[3a_5^2a_1 + 6a_5a_4a_2 + 3a_5a_3^2 + 3a_4^2a_3] \\
3a_5^2a_2 + 6a_5a_4a_3 + a_4^3 \\
-[3a_5^2a_3 + 3a_5a_4^2] \\
3a_5^2a_4 \\
-a_5^3
\end{array}$$

EXAMPLE $x^5 - 2x^4 + x^3 + x^2 - 3x - 2 = 0 \quad f(n) = n^2$

| | | PRODUCT |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 4 | 8 |
| 3 | 9 | 27 |
| 4 | 16 | 64 |
| 5 | 25 | 125 |
| 11 | 36 | 396 |
| 26 | 49 | 1274 |
| 54 | 64 | 3456 |
| 94 | 81 | 7614 |
| 151 | 100 | 15100 |
| 254 | 121 | 30734 |
| 477 | 144 | 68688 |
| 939 | 169 | 158691 |
| 1788 | 196 | 350448 |
| 3224 | 225 | 725400 |
| 5660 | 256 | 1448960 |
| 10079 | 289 | 2912831 |
| 18516 | 324 | 5999184 |

The recursion relation of the product corresponds to the equation:

$$x^{15} - 6x^{14} + 15x^{13} - 17x^{12} - 6x^{11} + 42x^{10} - 38x^9 - 21x^8 + 69x^7 - 17x^6 - 54x^5$$
$$+ 33x^4 + 21x^3 - 42x^2 - 36x - 8 = 0.$$

CHECK

6*725400 – 15*350448 + 17*158691 + 6*68688 – 42*30734 + 38*15100 + 21*7614 – 69*3456
+ 17*1274 + 54*396 – 33*125 – 21*64 + 42*27 + 36*8 + 8*1 = 1448960.

## ARITHMETIC PROGRESSION

An arithmetic progression is given by a polynomial function of the first degree so that its recursion relation corresponds to $(x - 1)^2 = 0$ with the root 1 taken twice. For a fifth order linear recursion relation such as given under the quadratic polynomial the coefficients of the equation corresponding to the linear recursion relation for the product are as follows.

$$1$$
$$-2a_1$$
$$2a_2 + a_1^2$$
$$-[2a_3 + 2a_2a_1]$$
$$2a_4 + 2a_3a_1 + a_2^2$$
$$-[2a_5 + 2a_4a_1 + 2a_3a_2]$$
$$2a_5a_1 + 2a_4a_2 + a_3^2$$
$$-[2a_5a_2 + 2a_4a_3]$$
$$2a_5a_3 + a_4^2$$
$$-2a_5a_4$$
$$a_5^2$$

## POLYNOMIAL OF THE THIRD DEGREE

For a third-degree polynomial and a recursion relation up to the third order the coefficients of the equation corresponding to the linear recursion relation of the product are given by the following.

$$1$$
$$-4a_1$$
$$4a_2+6a_1^2$$
$$-[4a_3+12a_2a_1+4a_1^3]$$
$$12a_3a_1+6a_2^2+12a_2a_1^2+a_1^4$$
$$-[12a_3a_2+12a_3a_1^2+12a_2^2a_1+4a_2a_1^3]$$
$$6a_3^2+24a_3a_2a_1+4a_3a_1^3+4a_2^3+6a_2^2a_1^2$$
$$-[12a_3^2a_1+12a_3a_2^2+12a_3a_2a_1^2+4a_2^3a_1]$$
$$12a_3^2a_2+6a_3^2a_1^2+12a_3a_2^2a_1+a_2^4$$
$$-[4a_3^3+12a_3^2a_2a_1+4a_3a_2^3]$$
$$4a_3^3a_1+6a_3^2a_2^2$$
$$-4a_3^3a_2$$
$$a_3^4$$

EXAMPLE

$$x^3-3x^2+x-2=0 \quad \text{and} \quad f(n)=n^3$$

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 8 | 16 |
| 3 | 27 | 81 |
| 9 | 64 | 576 |
| 28 | 125 | 3500 |
| 81 | 216 | 17496 |
| 233 | 343 | 79919 |
| 674 | 512 | 345088 |
| 1951 | 729 | 1422279 |
| 5645 | 1000 | 5645000 |
| 16332 | 1331 | 21737892 |
| 47253 | 1728 | 81653184 |
| 136717 | 2197 | 300367249 |
| 395562 | 2744 | 1085422128 |
| 1144475 | 3375 | 3862603125 |

The recursion relation of the product corresponds to the relation:

$$x^{12}-12x^{11}+58x^{10}-152x^9+267x^8-384x^7+442x^6-396x^5+337x^4-184x^3+120x^2$$
$$-32x+16=0.$$

CHECK

12*81653184 – 58*21737892 + 152*5645000 – 267*1422279 + 384*345088 – 442*79919 + 396*17496 – 337*3500 + 184*576 – 120*81 + 32*16 – 16 = 300367249

## REPEATED ROOTS IN GENERAL

Given a sequence whose recursion relation has $p$ repeated roots and another whose recursion relation has $q$ repeated roots. We would have:

$$S_n = r^n(a_0+a_1n+\cdots+a_{p-1}n^{p-1})$$
$$T_n = s^n(b_0+b_1n+\cdots+b_{q-1}n^{q-1})$$
$$Z_n = S_nT_n = (rs)^n(c_0+c_1n+\cdots+c_{p+q-2}n^{p+q-2})$$

so that the recursion relation of the product is of order $p+q-1$.

If the first recursion relation has $m$ distinct roots $r_i$ and a repeated root $r$ of multiplicity $p$, while the second has $n$ distinct roots $s_j$ and a repeated root of multiplicity $q$, the product recursion relation has the following number of roots: $mn+pn+qm+p+q-1=(m+p)(n+q)-(p-1)(q-1)$. The symmetric functions of these roots will give the coefficients of a recursion relation of this order. Similar considerations can be applied when there are a number of repeated roots of various multiplicities.

*******