ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico 87131

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S. E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \qquad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \qquad L_0 = 0, \qquad L_1 = 1.$$

Also a and b designate the roots of $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-328 Proposed by Walter Hansell, Mill Valley, California, and V. E. Hoggatt, Jr., San Jose, California.

Show that

$$6(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

is always a sum

$$m^2 + (m^2+1) + (m^2+2) + \cdots + (m^2+r)$$

of consecutive integers, of which the first is a perfect square.

B-329 Proposed by Herta T. Freitag, Roanoke, Virginia.

Find r, s, and t as linear functions of n such that $2F_r^2 - F_sF_t$ is an integral divisor of $L_{n+2} + L_n$ for $n = 1, 2, \cdots$.

B-330 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Let

$$G_n = F_n + 29F_{n+4} + F_{n+8}\,.$$

Find the greatest common divisor of the infinite set of integers $\{G_0, G_1, G_2, \cdots\}$.

B-331 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Prove that

$$F_{6n+1}^2 \equiv 1 \pmod{24}.$$

B-332 Proposed by Phil Mana, Albuquerque, New Mexico.

Let $a(n)$ be the number of ordered pairs of integers (r,s) with both $0 \leqslant r \leqslant s$ and $2r + s = n$. Find the generating function

$$A(x) = a(0) + xa(1) + x^2a(2) + \cdots.$$

B-333 Proposed by Phil Mana, Albuquerque, New Mexico.

Let S_n be the set of ordered pairs of integers (a,b) with both $0 < a < b$ and $a + b \leqslant n$. Let T_n be the set of ordered pairs of integers (c,d) with both $0 < c < d < n$ and $c + d > n$. For $n \geqslant 3$, establish at least one bijection (i.e., 1-to-1 correspondence) between S_n and T_{n+1}.

SOLUTIONS

SO BEE IT

B-304 Proposed by Sidney Kravitz, Dover, New Jersey.

According to W. Hope-Jones "The Bee and the Pentagon," *The Mathematical Gazette,* Vol. X, No. 150, 1921 (Reprinted Vol. LV, No. 392, March 1971, Page 220) the female bee has two parents but the male bee has a mother only. Prove that if we go back n generations for a female bee she will have F_n male ancestors in that generation and F_{n+1} female ancestors, making a total of F_{n+2} ancestors.

Solution by Sister Marion Beiter, Rosary Hill College, Buffalo, New York.

The proof is by induction. Let $P(n)$ be the statement of the problem. $P(n)$ holds for $n = 1$.

One generation back a female bee will have $F_1 = 1$ male ancestor and $F_{1+1} = 1$ female ancestor, a total of $F_{1+2} = 2$. If $P(n)$ holds for $n = k$, it holds for $n = k + 1$:

If we go back k generations for a female bee she will have F_k male ancestors in that generation and F_{k+1} female ancestors, making a total of F_{k+2} ancestors.

Then if we go back $k + 1$ generations, she will have F_{k+1} male ancestors (from the F_{k+1} females in the k^{th} generation), and F_{k+2} female ancestors (from the total F_{k+2} ancestors in the k^{th} generation). This makes a total of $F_{k+1} + F_{k+2} = F_{k+3}$. Hence, $P(n)$ holds for all natural numbers n.

Also solved by George Berzsenyi, Paul S. Bruckman, Herta T. Freitag, Graham Lord, A. G. Shannon, and the Proposer.

A TELESCOPING SUM

B-305 Proposed by Frank Higgins, North Central College, Naperville, Illinois.

Prove that

$$F_{8n} = L_{2n} \sum_{k=1}^{n} L_{2n+4k-2}\,.$$

Solution by Graham Lord, Universite Laval, Quebec, Canada.

The following steps use standard identities:

$$\begin{aligned} L_{2n} \sum_{k=1}^{n} L_{2n+4k-2} &= L_{2n} \left(\sum_{k=1}^{n} \left\{ F_{2n+4k-2+2} - F_{2n+4k-2-2} \right\} \right) \\ &= L_{2n}(F_{6n} - F_{2n}) = L_{2n} \cdot F_{2n} \cdot L_{4n} \\ &= F_{4n} \cdot L_{4n} = F_{8n}\,. \end{aligned}$$

Also solved by George Berzsenyi, Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, and the Proposer.

SOMETHING SPECIAL

B-306 Proposed by Frank Higgins, North Central College, Naperville, Illinois.

Prove that

$$F_{8n+1} - 1 = L_{2n} \sum_{k=1}^{n} L_{2n+4k-1}\,.$$

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

For generalized Fibonacci numbers defined by letting H_0 and H_1 be arbitrary integers and $H_n = H_{n-1} + H_{n-2}$ for $n \geqslant 2$, it is known that

$$\sum_{k=1}^{n} H_{4k-1} = F_{2n} H_{2n+1}\,.$$

(See, for example, Identity (9) in Iyer's article, *FQ*, 7 (1969), pp. 66–72.) More generally,

$$\sum_{k=1}^{n} H_{2n+4k-1} = F_{2n} H_{4n+1}\,.$$

Specializing this identity to Lucas numbers and using (I_7) and (I_{24}) of Hoggatt's *Fibonacci and Lucas Numbers,* one obtains

$$L_{2n}\sum_{k=1}^{n} L_{2n+4k-1} = L_{2n}F_{2n}L_{4n+1} = F_{4n}L_{4n+1} = F_{8n+1}-1.$$

Also solved be Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Graham Lord, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, and the Proposer.

MODULARLY MOVING MAVERICK

B-307 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Let

$$(1+x+x^2)^n = a_{n,0}+a_{n,1}x+a_{n,2}x^2+\cdots,$$

(where, of course, $a_{n,k}=0$ for $k>2n$). Also let

$$A_n = \sum_{j=0}^{\infty} a_{n,4j},\qquad B_n = \sum_{j=0}^{\infty} a_{n,4j+1},\qquad C_n = \sum_{j=0}^{\infty} a_{n,4j+2},\qquad D_n = \sum_{j=0}^{\infty} a_{n,4j+3}.$$

Find and prove the relationship of A_n, B_n, C_n and D_n to each other. In particular, show the relationship among these four sums for $n = 333$.

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

One may easily prove by induction on n that the trinomial coefficients $a_{n,k}$ satisfy the recursion formula

$$a_{n,k} = a_{n-1,k-2}+a_{n-1,k-1}+a_{n-1,k}$$

for $n>0$ with initial values

$$a_{0,k} = \begin{cases} 1 \text{ if } k=0\\ 0,\text{ otherwise}\end{cases}.$$

By letting $x = 1$ in the defining equation one may also deduce that

$$\sum_{k=0}^{\infty} a_{n,k} = 3^n.$$

This last fact and two applications of the recurrence relationship readily yield the following identities for $n \geqslant 2$:

$$A_n = 2\cdot 3^{n-2}+C_{n-2},\qquad C_n = 2\cdot 3^{n-2}+A_{n-2},$$
$$B_n = 2\cdot 3^{n-2}+D_{n-2},\qquad D_n = 2\cdot 3^{n-2}+B_{n-2}.$$

Iteration on n, upon summation of the resulting geometric series, yields the following formula for each

$$X\in\{A, B, C, D\},\qquad m\in\{0,1,2,3,\},\qquad n = 0,1,2,\cdots:$$

$$X_{4n+m} = \tfrac{1}{4}(3^{4n+m}-3^m)+X_m.$$

Less compactly, but more in the spirit of comparison one finds

$$B_{4n} = C_{4n} = D_{4n} = A_{4n}-1,\qquad B_{4n+1} = C_{4n+1} = A_{4n+1} = D_{4n+1}+1,$$
$$B_{4n+2} = D_{4n+2} = A_{4n+2} = C_{4n+2}-1,\qquad C_{4n+3} = D_{4n+3} = A_{4n+3} = B_{4n+3}+1,$$

for each $n = 0, 1, 2, \cdots$. In particular,

$$A_{333} = B_{333} = C_{333} = D_{333}+1 = \tfrac{1}{4}(3^{333}+1).$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, David Zeitlin, and the Proposer.

A GARBLED HINT

B-308 Proposed by Phil Mana, Albuquerque, New Mexico.

(b) Let r be a real number such that $\cos(r\pi) = p/q$, with p and q relatively prime positive integers and q not in $\{1, 2, 4, 8, \cdots\}$. Prove that r is not rational.

[The (a) part has been deleted due to an error in it.]

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Ill.

(b) We first recall the multiple-angle formula from trigonometry:

$$\cos n\theta = \tfrac{1}{2}\sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2\cos\theta)^{n-2k}, \qquad n = 1, 2, 3, \cdots. \tag{1}$$

We also recall, or we may easily show, that this is a polynomial with integer coefficients, in $\cos\theta$.

Suppose now that $r = u/v$ is rational (with u and v relatively prime natural numbers), and satisfies:

$$\cos(r\pi) = p/q, \tag{2}$$

where p and q are relatively prime natural numbers and $(q,2) = 1$ (i.e., q is odd), except $q \neq 1$. Note that this restricts q more than in the original problem, but we will deal with the remaining values of q later. Letting $\theta = r\pi$ and $n = v$ in (1), we get:

$$(-1)^u = \cos u\pi = \tfrac{1}{2}\sum_{k=0}^{[v/2]} (-1)^k \frac{v}{v-k}\binom{v-k}{k}\left(\frac{2p}{q}\right)^{v-2k} = \frac{2^{v-1}p^v}{q^v} - \frac{v2^{v-3}p^{v-2}}{q^{v-2}} + \cdots$$

$$= \frac{2^{v-1}p^v + q^2M}{q^v},$$

where M is some integer.

Since $(2,q) = (p,q) = 1$, it follows that $(2^{v-1}p^v + q^2M, q^v) = 1$; but then q^v cannot divide $(2^{v-1}p^v + q^2M)$, and their ratio cannot be $(-1)^u = \pm 1$. This contradiction shows that r cannot be rational, when q is as stated above.

Suppose now, as before, that (2) holds for some rational r, where $q = 2^s t$, $(2,p) = (p,t) = (2,t) = 1$, s and t are natural numbers, $t \geq 3$. As before,

$$(-1)^u = (2^{v-1}p^v + q^2M)/q^v = (2^{v-1}p^v + 2^{2s}t^2M)/2^{sv}t^v.$$

Since $(2,t) = (p,t) = 1$, the indicated ratio cannot be an integer, and we have again reached a contradiction. Hence, we have proved that the only possible values of q satisfying (2) are $q = 1$ and $q = 2$; this, in turn, implies that $\cos(r\pi) = 0, \pm\tfrac{1}{2}, \pm 1$ are the only possible values, corresponding to $r = n + \tfrac{1}{2}$, $n \pm 1/3$, and n, respectively, where n is an arbitrary integer. This is a stronger result that originally sought.

Also solved by the Proposer. The error in Part (a) was pointed out by Paul S. Bruckman and Herta T. Freitag.

AN ANALOGUE OF $a^n = aF_n + F_{n-1}$

B-309 Corrected version of B-284.

Let $z^2 = xz + y$ and let k, m, and n be nonnegative integers. Prove that

(a) $z^n = p_n(x,y)z + Q_n(x,y)$, where p_n and Q_n are polynomials in x and y with integer coefficients and p_n has degree $n - 1$ *in* x *for* n > 0.

(b) There are polynomials r, s, and t, not all identically zero and with integer coefficients, such that

$$z^k r(x,y) + z^m s(x,y) + z^n t(x,y) = 0.$$

Composite of solutions by David Zeitlin, Minneaspolis, Minnesota, and the Proposer.

(a) Let $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = xU_{n+1} + yU_n$ for $n = 0, 1, \cdots$. Then one easily proves by induction that $z^n = zU_n + U_{n-1}$ using $z^{k+1} = xz^k + yz^{k-1}$.

(b) z satisfies a quadratic equation over the field $F = Q(x,y)$ of polynomials in x and y with rational coefficients. Hence $F[z]$ is a vector space of degree 2 over F. Thus any three powers of z are linearly dependent over F. Clearing denominators, gives the desired result.

Also solved by Paul S. Bruckman and Herta T. Freitag.

****