PASCAL, CATALAN, AND GENERAL SEQUENCE CONVOLUTION ARRAYS IN A MATRIX

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The Catalan numbers $\{1, 1, 2, 5, 14, 42, \cdots\}$ are the first sequence in a sequence of sequences S_i which arise in the first column of matrix inverses of matrices containing certain columns of Pascal's triangle, and which also can be obtained from certain diagonals of Pascal's triangle [1], [2]. These sequences S_i are also the solutions for certain ballot problems, which counting process also yields their convolution arrays. The convolution triangles for the sequences S_i contain determinants with special values and occur in matrix products yielding Pascal's triangle. Surprisingly enough, we can also find determinant properties which hold for any convolution array.

1. ON THE CATALAN NUMBERS AND BALLOT PROBLEMS

When the central elements of the even rows of Pascal's triangle are divided sequentially by 1, 2, 3, 4, ⋯, to obtain 1/1 = 1, 2/2 = 1, 6/3 = 2, 20/4 = 5, 70/5 = 14, 252/6 = 42, ⋯, the Catalan sequence $\{C_n\}$ results,

(1.1) $$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n = 0, 1, 2, \cdots.$$

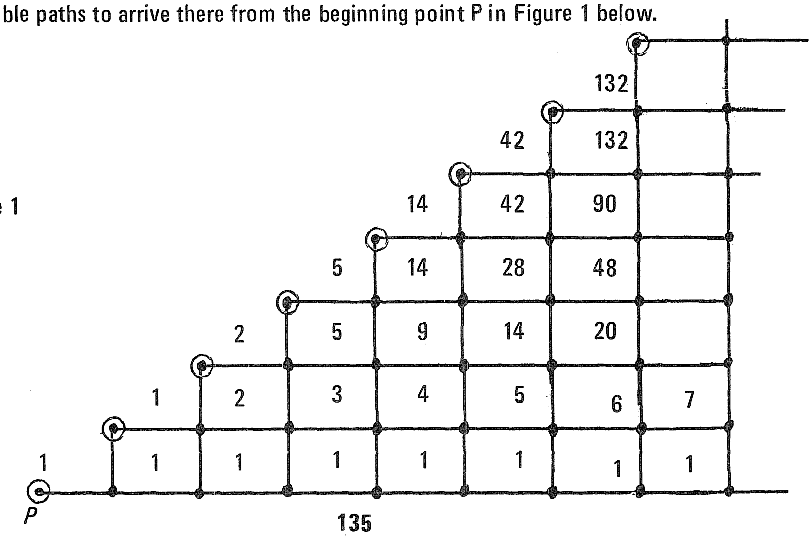
The Catalan sequence has the generating function [4]

(1.2) $$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n$$

and appears in several ways in Pascal's triangle.

The Catalan numbers also arise as the solution to a counting problem, being the number of paths possible to travel from a point to points lying along a rising diagonal, where one is allowed to travel from point to point within the array by making one move to the right horizontally or one move vertically. Each point in the array is marked with the number of possible paths to arrive there from the beginning point P in Figure 1 below.

Figure 1



Given two sequences $\{a_n\}$ and $\{b_n\}$, there is a sequence $\{c_n\}$ called the *convolution* of the two sequences,

$$c_n = \sum_{k=0}^{n} b_{n-k}a_k .$$

If the sequences have generating functions $A(x)$, $B(x)$, and $C(x)$, respectively, then $C(x) = A(x)B(x)$. The successive convolutions of the Catalan sequence with itself appear as successive columns in the *convolution triangle*

1							
1	1						
2	2	1					
5	5	3	1				
14	14	9	4	1			
42	42	28	14	5	1		
132	132	90	48	20	6	1	
...	...	...	...	...	...	...	...

Notice that these same sequences appear on successive diagonals in Figure 1.

Call the Catalan sequence S_1, the first of a sequence of sequences S_i which arise as the solutions to similar counting problems where one changes the array of points. The counting problem related to S_2 we illustrate in Figure 2 below. The circled vertices yield $S2 = \{1, 1, 3, 12, 55, 273, \cdots\}$; under this is the first convolution $\{1, 2, 7, 30, 143, \cdots\}$, which can be computed from the definition of convolution. Successive diagonals continue to give successive convolutions of S_2.

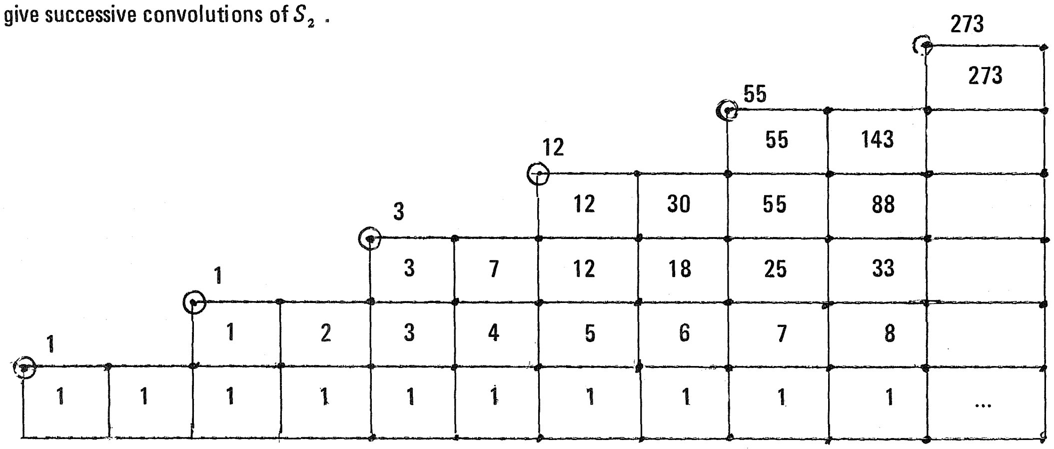


Figure 2

Similarly for $S_3 = \{1, 1, 4, 22, 140, \cdots\}$, the circled vertices are the sequence S_3, and under this appears the first convolution, and so on, as shown in Figure 3.

The sequences S_i and their convolution triangles are the solution to such counting problems, where one counts the number of paths possible to arrive at each point in the array from a beginning point from which one is allowed to travel from point to point within the array by making one move to the right horizontally or one move vertically. For the sequence S_i, the points in the grid are arranged so that the successive circled points are i to the right and one above their predecessors. By the rule of formation as compared to the rule of formation of the convolution array for S_i as found in [1], one sees that we have the same sequences S_i in both cases. Here, we go on to relate these convolution arrays to Pascal's triangle as matrix products.

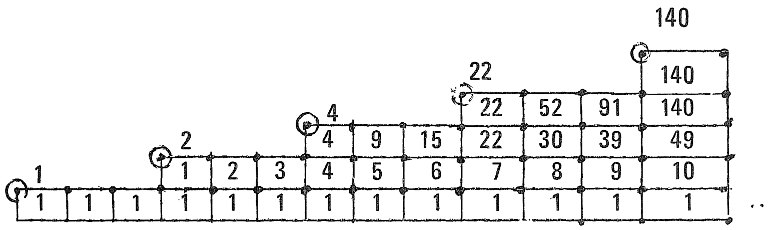


Figure 3

2. THE CATALAN CONVOLUTION TRIANGLE IN A MATRIX

Write a matrix A which contains the rows of Pascal's triangle from Fig. 1 written on and below the main diagonal with alternating signs. Write a matrix B containing the Catalan convolution triangle on and below its main diagonal, augmented by the first column of the identity matrix on the left. Then B is the matrix inverse of A, so that $AB = I$, the identity matrix, where, of course, all matrices have the same order. That is, for order 7,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 & 1 & 0 \\ 0 & 0 & 0 & -1 & 6 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 3 & 1 & 0 & 0 \\ 0 & 14 & 14 & 9 & 4 & 1 & 0 \\ 0 & 42 & 42 & 28 & 14 & 5 & 1 \end{bmatrix} = I .$$

As proof, the columns of A are generated by $[x(1-x)]^{j-1}$ while those of B are generated by $[(1-\sqrt{1-4x})/2]^{j-1}$. The columns of AB, then, are the composition

$$[(1-\sqrt{1-4(1-x)x})/2]^{j-1} = x^{j-1} ,$$

the column generators for the identity matrix. Notice that the row sums of the absolute values of the elements of A are the Fibonacci numbers, 1, 1, 2, 3, 5, 8, 13, ⋯, while the row sums of B are the Catalan numbers.

Now, if the Catalan convolution triangle is written as a square array and used to form a matrix C, and if Pascal's triangle is written as a square array to form matrix P, then P is the matrix product AC. First, we illustrate for 5×5 matrices A, C, and P:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 9 & 14 & 20 \\ 5 & 14 & 28 & 48 & 75 \\ 14 & 42 & 90 & 165 & 275 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix} .$$

Again considering the column generators and finding their composition, we prove that $AC = P$. The column generators of C are $[(1-\sqrt{1-4x})/2x]^{j-1}$, making the column generators of the matrix product AC to be

$$[(1-\sqrt{1-4x(1-x)})/2x(1-x)]^{j-1} = [1/(1-x)]^{j-1} ,$$

the generating functions of the columns of Pascal's triangle written in the form of P.

Fortunately, the finite $n \times n$ lower left matrices A have determinants whose values are determined by an $n \times n$ determinant within the infinite one. For infinite matrices A, B, and C, if we know that $AB = C$ by generating functions, then it must follow that $AB = C$ for $n \times n$ matrices, A, B, and C, because each $n \times n$ matrix is the same as the $n \times n$ block in the upper left in the respective infinite matrix. That is, adding rows and columns to the $n \times n$ matrices A and B does not alter the minor determinants we had, and similarly, the $n \times n$ matrix C agrees with the infinite matrix C in its $n \times n$ upper left corner. We write the Lemma,

Lemma. Let A be an infinite matrix such that all of its non-zero elements appear on and below its main diagonal, and let $A_{n\times n}$ be the $n \times n$ matrix formed from the upper left corner of A. Let B and C be infinite matrices with $B_{n\times n}$ and $C_{n\times n}$ the $n \times n$ matrices formed from their respective upper left corners. If $AB = C$, then $A_{n\times n}B_{n\times n} = C_{n\times n}$.

We will frequently consider $n \times n$ submatrices of infinite matrices in this paper, but we will not describe the details above in each instance. We can apply earlier results [2], [3] to state the following theorems for the Catalan convolution array, since each submatrix of C in Theorem 2.1 and 2.2 is multiplied by a submatrix of A which has a unit determinant to form the similarly placed submatrix within Pascal's triangle written in rectangular form.

Theorem 2.1. The determinant of any $n \times n$ array taken with its first row along the row of ones in the Catalan convolution array written in rectangular form is one.

Theorem 2.2. The determinant of any $k \times k$ array taken from the Catalan convolution array written in rectangular form with its first row along the second row of the Catalan convolution array and its first column the j^{th} column of the array has its value given by the binomial coefficient

$$\binom{k+j-1}{k}.$$

On the other hand, taking alternate columns of Pascal's triangle with alternating signs to form matrix Q and alternate columns of the Catalan convolution triangle to form matrix R as indicated below produces a pair of matrix inverses, where the row sums of absolute values of the elements of Q are the alternate Fibonacci numbers 1, 2, 5, 13, 34, ···, F_{2k+1}, ···, while the row sums of R are

$$1, 2, 6, 20, 70, \cdots, \binom{2n}{n}, \cdots,$$

the central column of Pascal's triangle. For 6×6 matrices Q and R,

$$QR = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ -1 & 6 & -5 & 1 & 0 & 0 \\ 1 & -10 & 15 & -7 & 1 & 0 \\ -1 & 15 & -35 & 28 & -9 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 5 & 9 & 5 & 1 & 0 & 0 \\ 14 & 28 & 20 & 7 & 1 & 0 \\ 42 & 90 & 75 & 35 & 9 & 1 \end{bmatrix} = I.$$

Here, the j^{th} column of Q is generated by

$$Q(x) = \frac{1}{1+x} \cdot \left(\frac{x}{(1+x)^2}\right)^{j-1}$$

while the j^{th} column of R is generated by

$$R(x) = \frac{1-\sqrt{1-4x}}{2x} \cdot \left(\frac{(1-\sqrt{1-4x})^2}{4x}\right)^{j-1}$$

so that the j^{th} column of QR is generated by

$$\left(\frac{1-\left(1-\frac{4x}{(1+x)^2}\right)^{2}}{\frac{4x}{(1+x)^2}}\right)^{j-1} = \left(\frac{\left(1-\frac{1-x}{1+x}\right)^{2}}{\frac{4x}{(1+x)^2}}\right)^{j-1} = \left(\frac{\frac{4x^2}{(1+x)^2}}{\frac{4x}{(1+x)^2}}\right)^{j-1} = x^{j-1}$$

the generating function for the identity matrix.

3. MATRICES FORMED FROM CONVOLUTION TRIANGLES OF THE SEQUENCES S_i

Now we generalize, applying similar thinking to the sequences S_i. We use the notation of [2], letting $P_{i,j}$ be the infinite matrix formed by placing every j^{th} column (beginning with the zeroth column which contains the sequence S_i^1) of the convolution triangle for the sequence S_i on and below its main diagonal, and zeroes elsewhere. Then $P_{0,j}$ contains every j^{th} column of the convolution array for $S_0 = \{1, 1, 1, \cdots\}$, which is Pascal's triangle. Let $P'_{i,j}$ denote the matrix formed as $P_{i,j}$ from every j^{th} column of the convolution array for S_i but beginning with the column which contains S_i^0, so that the n^{th} column of the matrix contains the sequence $S_i^{(n-1)j}$. Let $P^*_{i,j}$ be formed as $P_{i,j}$ except that the columns are written in a rectangular display, so that the first row is a row of ones. Then, referring to Section 2,

$$A = P'^{-1}_{1,1}, \quad B = P'_{1,1}, \quad C = P^*_{1,1}, \quad P = P^*_{0,1}, \quad R = P_{1,2},$$

so that $AC = P$ becomes

(3.1) $$P'^{-1}_{1,1}P^*_{1,1} = P^*_{0,1}.$$

One also finds that

(3.2) $$P^{-1}_{1,1}P^*_{1,1} = P'^*_{0,1}.$$

We extend these results to S_2, where we will illustrate first $P^{-1}_{2,2}P_{2,2} = I$ for 5 × 5 submatrices [see 2]:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & 1 & 0 \\ 0 & -1 & 10 & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 \\ 12 & 12 & 5 & 1 & 0 \\ 55 & 55 & 25 & 7 & 1 \end{bmatrix} = I$$

Here, the row sums of $P_{2,2}$ are 1, 2, 7, 30, 143, ···, which we recognize as S_2^2, the first convolution of S_2. Notice that $P^{-1}_{2,2}$ contains the odd rows of Pascal's triangle as its columns.

If we form $P'^{-1}_{2,2}$ using the even rows of Pascal's triangle taken with alternate signs on and below the main diagonal, then

(3.3) $$P'^{-1}_{2,2}P^*_{2,1} = P^*_{0,1},$$

which we illustrate for 5 × 5 matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 6 & -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 7 & 12 & 18 & 25 \\ 12 & 30 & 55 & 88 & 130 \\ 55 & 143 & 273 & 455 & 700 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}.$$

Of course, this means that the results of Theorems 2.1 and 2.2 also apply for the sequence S_2.

Now, if the matrix $P^*_{2,2}$ is formed from every other column of the S_2 convolution array written in rectangular form, the matrix product $P'^{-1}_{2,2}P^*_{2,2}$ becomes the matrix containing every other column of Pascal's triangle written in rectangular form. For example, for the 5 × 5 case,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 6 & -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 & 9 \\ 3 & 12 & 25 & 42 & 63 \\ 12 & 55 & 130 & 245 & 408 \\ 55 & 273 & 700 & 1428 & 2565 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 & 9 \\ 1 & 6 & 15 & 28 & 45 \\ 1 & 10 & 35 & 84 & 165 \\ 1 & 15 & 70 & 210 & 495 \end{bmatrix}.$$

This also means that, using earlier results [3], if we take the determinant of any square submatrix of $P^*_{2,2}$ with its first row taken along the first row of $P^*_{2,2}$ the determinant value will be $2^{[k(k-1)/2]}$ if the submatrix taken has order k.

If we shift the columns of $P^*_{2,1}$ one to the left so that the new matrix begins with S_2^2 in its first column, we find that, for 5 × 5 submatrices,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 3 & -5 & 1 & 0 \\ 0 & -1 & 10 & -7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 7 & 12 & 18 & 25 & 33 \\ 30 & 55 & 88 & 130 & 182 \\ 143 & 273 & 455 & 700 & 1020 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}$$

We shall show that this is also true for the infinite matrices indicated, which means, in light of our previous results, that the results of Theorems 2.1 and 2.2 also apply to the rectangular convolution array for S_2 if we truncate its zero$^{\text{th}}$ column.

Using every other column, we can make some interesting shifts. We already observed that $P'^{-1}_{2,2}P^*_{2,2} = P^*_{0,2}$. We also can write $P^{-1}_{2,2}P^*_{2,2}$, which provides every other column of Pascal's triangle, beginning with the column of integers. We also can write two matrix products relating the matrix containing the odd columns of the convolution matrix for S_2 to matrices containing every other column of Pascal's triangle, each of which is illustrated below for 4×4 submatrices.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 3 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 7 & 18 & 33 & 52 \\ 30 & 88 & 182 & 320 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 6 & 15 & 28 \\ 1 & 10 & 35 & 84 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 7 & 18 & 33 & 52 \\ 30 & 88 & 182 & 320 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \\ 3 & 10 & 21 & 36 \\ 4 & 20 & 56 & 120 \end{bmatrix}.$$

Since we can establish that the corresponding infinite matrices do have the product indicated, if we form a rectangular array from the convolution array for S_2 using every other column, whether we take the odd columns only, or the even columns only, the determinant of any $k \times k$ submatrix of either array which has its first row taken along the first row of the array will have determinant value given by $2^{[k(k-1)/2]}$.

Next, form $P_{3,3}$ containing every third column of the convolution triangle for S_3. Then, from [2], $P^{-1}_{3,3}$ contains every third row of Pascal's triangle taken with alternate signs on and below the main diagonal with zeroes elsewhere, as illustrated for 5 × 5 submatrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 & 0 \\ 0 & 6 & -7 & 1 & 0 \\ 0 & -4 & 28 & -10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 \\ 22 & 22 & 7 & 1 & 0 \\ 140 & 140 & 49 & 10 & 1 \end{bmatrix} = I,$$

Notice that the row sums of $P_{3,3}$ are 1, 2, 9, 52, 340, ···, or S_3^2, the first convolution of S_3. As before, we find that

$$P'^{-1}_{3,3}P^*_{3,1} = P^*_{0.1}, \tag{3.4}$$

which allows us to again extend Theorems 2.1 and 2.2. We also find

$$P'^{-1}_{3,3}P^*_{3,3} = P^*_{0,3} \tag{3.5}$$

which is illustrated for 5 × 5 submatrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 3 & -6 & 1 & 0 \\ 0 & -1 & 15 & -9 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 7 & 10 & 13 \\ 4 & 22 & 49 & 85 & 130 \\ 22 & 140 & 357 & 700 & 1196 \\ 140 & 969 & 2695 & 5740 & 10647 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 7 & 10 & 13 \\ 1 & 10 & 28 & 55 & 91 \\ 1 & 20 & 84 & 220 & 455 \\ 1 & 35 & 210 & 715 & 1820 \end{bmatrix}$$

Using earlier results [3], this means that, if we take any $k \times k$ submatrix of $P^*_{3,3}$ which has its first row along the first row of $P^*_{3,3}$, the value of its determinant is $3^{[k(k-1)/2]}$. However, we have the same result if we take every third column to form the array, whether we take columns of the convolution array for S_3 of the form $3k$, $3k+1$, or $3k+2$.

Next, we summarize our results. First, the matrix $P^{-1}_{i,i}$ always contains the i^{th} rows of Pascal's triangle written on and below the main diagonal with alternating signs, beginning with the first row, and zeroes elsewhere. That is, the j^{th} column of $P^{-1}_{i,i}$ contains the coefficients of $(1-x)^{1+i(j-1)}$, $j = 1, 2, \cdots$, on and below the main diagonal, and zeroes above the main diagonal. Inspecting $P_{i,i}$ gave the row sums as S_i^2, the first convolution of S_i. Both of these results were proved in [2].

If we form the matrix $P'^{-1}_{i,i}$ using the i^{th} rows of Pascal's triangle taken with alternate signs on and below the main diagonal, but beginning with the zero$^{\text{th}}$ row, so that the j^{th} column contains the coefficients of $(1-x)^{i(j-1)}$, $j = 1, 2, \cdots$, and form the matrix $P^*_{i,1}$ so that its elements are the convolution triangle for S_i written in rectangular form, then

$$P'^{-1}_{i,i}P^*_{i,1} = P^*_{0,1}, \tag{3.6}$$

the matrix containing Pascal's triangle written in rectangular form.

If we form an infinite matrix $P^*_{i,i}$ from every i^{th} column of the convolution array for the sequence S_i, then the matrix product

$$P'^{-1}_{i,i}P^*_{i,i} = P^*_{0,i} \tag{3.7}$$

the matrix formed from every i^{th} column of Pascal's triangle written in rectangular form. Further, (3.7) is only one of $2i$ similar matrix products which we could write. By adjusting the columns of $P^{-1}_{i,i}$ to write modified matrices which are formed using the i^{th} rows of Pascal's triangle as before but taking the first column of the new matrix $(P^{-1}_{i,i})_r$ to contain the r^{th} row, so that its j^{th} column contains the $(r+(j-1)i)$ row of Pascal's triangle or the coefficients of $(1-x)^{r+i(j-1)}$, $j = 1, 2, \cdots$, on and below its main diagonal, we can write

$$(P^{-1}_{i,i})_r P^*_{i,i} = (P^*_{0,i})_r, \qquad r = 0, 1, \cdots, i-1, \tag{3.8}$$

where $(P^*_{0,i})_r$ contains every i^{th} column of Pascal's triangle written in rectangular form beginning with its r^{th} column. Notice that $r = 0$ in (3.8) gives (3.7), and that

$$P'^{-1}_{i,i} = (P^{-1}_{i,i})_0 \qquad \text{while} \qquad P^{-1}_{i,i} = (P^{-1}_{i,i})_1 .$$

We can also write

$$P^{-1}_{i,i}(P^*_{i,i})_r = (P^*_{0,i})_{r-1}, \qquad r = 0, 1, \cdots, i-1, \tag{3.9}$$

where $(P^*_{i,i})_r$ contains the i^{th} columns of the convolution array for S_i beginning with the r^{th} column. Also,

$$(P^{-1}_{i,i})_r(P^*_{i,i})_r = (P^*_{0,i})_0 = P^*_{0,i}, \qquad r = 0, 1, \cdots, i-1 \tag{3.10}$$

$$(P^{-1}_{i,i})_j(P^*_{i,i})_r = (P^*_{0,i})_{r-j}. \tag{3.11}$$

The matrix identities of this section are proved next.

4. PROOF OF THE MATRIX IDENTITIES GIVEN IN SECTION 3

The proof of (3.3) follows from [2] but is a little subtle since we do not have explicit formulas for the generating functions for S_i, $i \geq 2$. However, we do have the following from [2]: If $S_i(x)$ is the generating function for S_i and if $S_0(x) = f(x)$, then

$$f(xS_1(x)) = S_1(x); \qquad f(S_2^2(x)) = S_2(x); \cdots; \qquad f(xS_k^k(x)) = S_k(x).$$

And further, $f(1/S_{-1}(x)) = S_1(x)$, etc. That means

$$S_2(x(1-x)^2) = S_2(x/[1/(1-x)^2]) = S_0(x) = \frac{1}{1-x}$$

which generates Pascal's triangle.

The general case given in (3.6) follows easily by replacing 2 with i in the above discussion. We prove (3.7) by taking

$$S_i(x[(1-x)^i]^i = \left(\frac{1}{1-x}\right)^i$$

the generating function for the matrix containing the i^{th} columns of Pascal's triangle. Equation (3.8) merely starts the matrices with shifted first columns, but it is the constant difference of the columns, or the power of $(1-x)$ which is the ratio of two successive column generators, which is used in the relationships shown above.

All of this raises a very interesting situation. Clearly, if we can obtain Pascal's triangle from the convolution array for S_i by matrix multiplication, then we can get the convolution array for any S_k by multiplying the convolution array for S_i by a suitable matrix. The possibilities are endless. Also, one can factor Pascal's triangle matrix when written in its rectangular form into several factors.

Now, in all of these special matrix multiplications, when $AB = C$, the column arrangement of B determines the column configuration of C. Whatever appears in A for i^{th} columns of a convolution array for S_i will appear as the rectangular convolution array for S_i (every column) if the proper middle matrix is used. Starting with, say, $S_2^2(x)$ as the first column of A and then $xS_2^5(x)$, $x^2S_2^8(x)$, $\cdots$, one can use as the middle matrix the one with column generators $(1+x)$, $(1+x)^2$, $(1+x)^3$, $\cdots$, where $S_{-1}(x) = (1+x)$. Now $S_{-1}(xS_2^3(x)) = S_2(x)$, etc. Thus the columns of the rightmost matrix are S_2^3, S_2^6, S_2^9, $\cdots$, as is to be expected.

5. DETERMINANT IDENTITIES IN CONVOLUTION ARRAYS

Since, in Section 3, we found several ways that $P_{i,1}^*$ and $P_{i,i}^*$, when multiplied by matrices having unit determinants, yield matrices containing columns of Pascal's triangle, and since the $n \times n$ submatrices taken in the upper left corners have the same multiplication properties as the infinite matrices from which they are taken in these cases, we have several theorems we can write by applying earlier results concerning determinant values found within Pascal's triangle [3]. Specifically, (3.5) and (3.7) allow us to write the very general theorem,

Theorem 5.1. Write the convolution array in rectangular form for any of the sequences S_i. Any $n \times n$ submatrix of the array which has its first row taken along the row of ones of the array has a determinant with value one. Any $n \times n$ submatrix of the array such that its first column lies in the j^{th} column of the array and its first row is taken along the row of integers of the array has determinant value given by the binomial coefficient $\binom{n+j-1}{n}$. Any $n \times n$ matrix formed such that its columns are every r^{th} column of the convolution array beginning with the j^{th} column, $j = 0, 1, \cdots, r-1$, has a determinant value of $r^{n(n-1)/2}$.

However, the surprising thing about Theorem 5.1 is that so much of it can be stated for the convolution array of *any* sequence whatever! Hoggatt and Bergum [5] have found that if S is any sequence with first term 1, then the rows of its convolution array written in rectangular form are arithmetic progressions of order 0, 1, 2, 3, $\cdots$ with constants 1, s_2, s_2^2, s_2^3, $\cdots$, where s_2 is the second term of sequence S. Applying Eves' Theorem [3],

Theorem 5.2. Let S be a sequence with first term one. If any $n \times n$ array is taken from successive rows and columns of the rectangular convolution array for S such that the first row includes the row of ones, then the determinant has value one if the second term of the sequence is one and value $s_2^{n(n-1)/2}$ if the second term of S is s_2.

Theorem 5.3. Let S be a sequence with first and second term both one. If any $n \times n$ array is formed from successive rows and columns of the rectangular convolution array for S such that the first row includes the row of integers and the first column includes S^{j-1}, $j = 1, 2, \cdots$, then the determinant of the array is given by the binomial coefficient $\binom{n+j-1}{n}$.

Conjecture. Let S be a sequence with first term one and second term s_2. If any $n \times n$ array is formed using the successive rows and columns of the rectangular convolution array for S such that the first row includes the row $1u_2$, $2u_2$, $3u_2$, $4u_2$, $\cdots$, and the first column includes S^{j-1}, $j = 1, 2, \cdots$, then the determinant of the array is given by

$$s_2^{n(n-1)/2}\binom{n+j-1}{n}.$$

For further interesting relationships, see Hoggatt and Bruckman [1].

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3. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Special Determinants Found within Generalized Pascal Triangles," *The Fibonacci Quarterly,* Vol. 11, No. 5 (Dec. 1973), pp. 457–465.
4. (Catalan thesis)
5. V. E. Hoggatt, Jr., and G. E. Bergum, "Generalized Convolution Arrays,"
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LETTER TO THE EDITOR

April 21, 1975

Dear Editor:

Following are some remarks on some formulas of Trumper [5].

Trumper has proved seven formulas of which the following is entirely characteristic

$$F_nF_m - F_xF_{n+m-x} = (-1)^{m+1}F_{x-m}F_{n-x}\,. \tag{1}$$

He actually gives 13 formulas, but the duplicity arises from the trivial replacement of x by $-x$ in all but the seventh formula.

It is of interest to note that the formulas are not really new in the sense that they can all be gotten from the single formula

$$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^nF_nF_b \tag{2}$$

by use of the negative transformation

$$F_{-n} = (-1)^{n+1}F_n\,. \tag{3}$$

For example, in (1) replace n by $n+x$ and m by $m+x$, and we have

$$F_{x+n}F_{x+m} - F_xF_{x+n+m} = (-1)^{m+x+1}F_{-m}F_n = (-1)^xF_mF_n\,,$$

the last step following by (3). But the formula is then simply a restatement of (2) with n replaced by x, a by n, and b by m. Similarly, for his formula (4), which we may rewrite as

$$F_{n+x}F_m - F_nF_{m+x} = (-1)^{m+1}F_{n-m}F_x\,,$$

we have only to set $x=a$, $m=n+b$ and use (3) again to get (2), and all steps are reversible. The reader may similarly derive the other formulas.

For reference to the history of (2), see [1, p. 404], [2], [3]. Formula (2) was posed as a problem [6]. Tagiuri is the oldest reference [4] of which I know. Formula (2) is the unifying theme behind all the formulas in [5].

[Continued on page 146.]