FIBONACCI TRIANGLE

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1. DEFINITION

The Fibonacci sequence $\{f_n\}$ is defined by the recursive relation

(1.1) $$f_n = f_{n-1} + f_{n-2}$$

with

(1.2) $$f_0 = f_1 = 1.^*$$

Let us define the set of integers $\{f_{m,n}\}$ with two suffices

(1.3) $$f_{m,n} = f_{m-1,n} + f_{m-2,n}$$

(1.4) $$f_{m,n} = f_{m-1,n-1} + f_{m-2,n-2}$$

$$(m \geqslant 2,\ m \geqslant n \geqslant 0)$$

with

(1.5) $$f_{0,0} = f_{1,0} = f_{1,1} = f_{2,1} = 1.$$

These numbers can be arranged triangularly as in Fig. 1,

1
1 1
2 1 2
3 2 2 3
5 3 4 3 5
8 5 6 6 5 8
13 8 10 9 10 8 13
21 13 16 15 15 16 13 21
34 21 26 24 25 24 26 21 34
55 34 42 39 40 40 39 42 34 55

Figure 1. Fibonacci Triangle

*Another set of the initial values can be chosen as

(1.2′) $$f_0 = 0, \qquad f_1 = 1.$$

where the entries in a row have the same m and line up according to the value of n $(0\text{-}m)$ from left to right.

Let us call them the Fibonacci Triangle.* A number of interesting relations were found, part of which will be given in this paper.

2. RELATION WITH THE FIBONACCI NUMBERS

As evident from the definitions (1.3)–(1.5) we have four Fibonacci sequences in the Triangle;

(2.1) $$f_{m,0} = f_{m,m} = f_m$$

(2.2) $$f_{m,1} = f_{m,m-1} = f_{m-1}.$$

Successive application of (1.3) to itself gives the following relation

(2.3) $$f_{m,n} = f_k \cdot f_{m-k,n} + f_{k-1} \cdot f_{m-k-1,n} \qquad (1 \leqslant k \leqslant m-n-1).$$

By putting $k = m-n-1$ into (2.3) one gets

(2.4) $$f_{m,n} = f_{m-n-1} \cdot f_{n+1,n} + f_{m-n-2} \cdot f_{n,n}$$

(from (2.1) and (2.2))

$$= f_n(f_{m-n-1} + f_{m-n-2}).$$

It follows then that

(2.5) $$f_{m,n} = f_{m-n} \cdot f_n \qquad (m \geqslant n \geqslant 0).$$

It means that the Fibonacci Triangle is constructed by the self-multiplication of the Fibonacci sequence, or symbolically,

(2.6) $$\{f_{m,n}\} = \{f_m\} \times \{f_n\}.$$

In other words the Fibonacci Triangle is the 2-dimensional Fibonacci sequence. Then extension to the k-dimensional Fibonacci sequence

(2.7) $$\{f_{m_1,m_2,\cdots,m_k}\} = \{f_{m_1}\} \times \{f_{m_2}\} \times \cdots \times \{f_{m_k}\}$$

is straightforward, but we will not duscuss them further.

It is proved from (2.5) or (2.6) that the Fibonacci sequences multiplied by the Fibonacci numbers are seen in the Triangle alongside of the "roof." That the Triangle is symmetric,

(2.8) $$f_{m,n} = f_{m,m-n},$$

is directly proved from (2.5). On the center line of the Triangle the squares of $\{f_n\}$ are lined up,

(2.9) $$f_{m,m/2} = (f_{m/2})^2 \qquad (m = \text{even}).$$

Application of the Binet's formula

(2.10) $$f_n = (\alpha^{n+1} - \beta^{n+1})/\sqrt{5}$$

$$\alpha = (1+\sqrt{5})/2, \qquad \beta = (1-\sqrt{5})/2$$

to (2.5) gives

(2.11) $$f_{m,n} = \{(\alpha^{m+2} + \beta^{m+2}) + (-1)^n(\alpha^{m-2n} + \beta^{m-2n})\}/5.$$

The Lucas sequence $\{g_n\}$, which is defined by

(2.12) $$g_n = g_{n-1} + g_{n-2}$$

with

(2.13) $$g_0 = 2, \qquad g_1 = 1,$$

is expressed as

(2.14) $$g_n = \alpha^n + \beta^n.$$

Thus one gets

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*The author noticed that the term "Fibonacci Triangle" is used for quite a different array of integers. [1].

(2.15) $$f_{m,n} = \left\{ g_{m+2} + (-1)^n g_{m-2n} \right\} /5$$

(2.16) $$= \left\{ g_{m+2} + (-1)^{m-n} q_{2n-m} \right\} /5.$$

In deriving (2.16) the following relation

(2.17) $$g_{-n} = (-1)^n g_n$$

was used.

3. MAGIC DIAMOND

As all the entries in the Triangle are generated from four one's (1.5) forming a diamond, any quartet $(f_{m,n}, f_{m-1,n}, f_{m-1,n-1}, f_{m-2,n-1})$ in the Triangle generates the nearest neighbors to the four corners and will be called as a "magic diamond."

Application of (1.3) into (1.4) gives "downward generation"

(3.1) $$f_{m,n} + f_{m-1,n} + f_{m-1,n-1} + f_{m-2,n-1} = f_{m+2,n+1} ,$$

as illustrated in Fig. 2a. Similarly one gets "upward generation"

(3.2) $$f_{m,n} - f_{m-1,n} - f_{m-1,n-1} + f_{m-2,n-1} = f_{m-4,n-2}$$

as in Fig. 2b.

"Leftward generation" and "rightward generation" (Figs. 2c,d) are obtained respectively as

(3.3) $$f_{m,n} + f_{m-1,n} - f_{m-1,n-1} - f_{m-2,n-1} = f_{m-1,n-2}$$

and

(3.4) $$f_{m,n} - f_{m-1,n} + f_{m-1,n-1} - f_{m-2,n-1} = f_{m-1,n+1}.$$

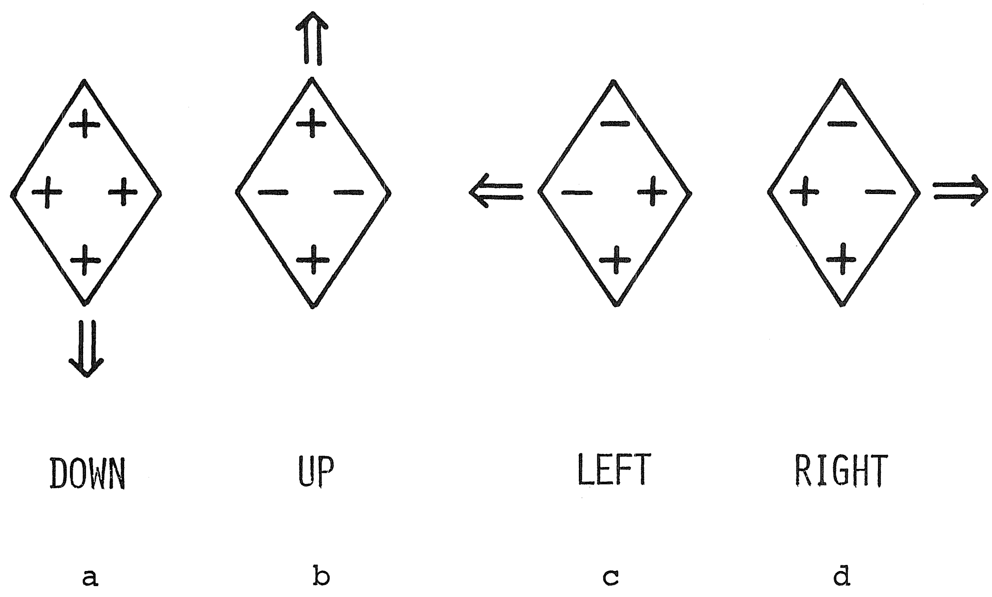

Figure 2. Magic Diamond

From (2.5) one gets

(3.5) $$f_{m,n} \cdot f_{m-2,n-1} = f_{m-1,n-1} \cdot f_{m-1,n}$$

or

(3.6) $$f_{m,n} \div f_{m-1,n} \div f_{m-1,n-1} \times f_{m-2,n-1} = 1,$$

which shows the stability of the "magic diamond" (see Fig. 3).

It is verified from (2.5) that the four numbers at the corners in any parallelogram are stable like an "Amoeba."

(3.7) $$f_{m,n} \cdot f_{m-k-\ell,n-k} = f_{m-k,n-k} \cdot f_{m-\ell,n} \cdot$$

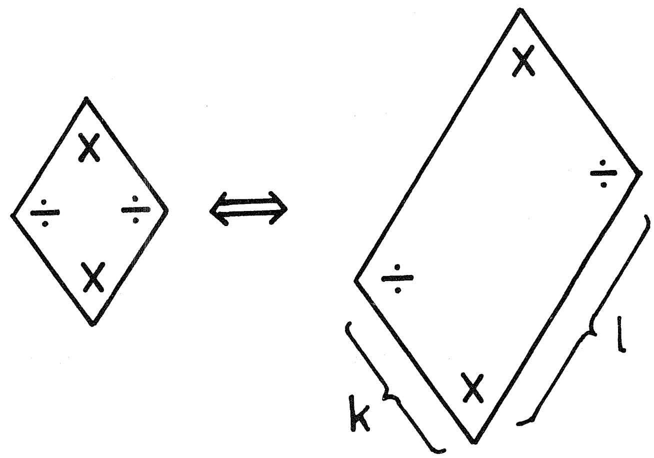

Figure 3 Amoeba

4. CRAWLING CRAB

The sum of the three entries in any downward triangle $(f_{m,n}, f_{m-1,n}, f_{m-1,n-1})$ or a "Crab" is kept constant as long as the Crab crawls sideways (see Fig. 4a),

(4.1) $$f_{m,n} + f_{m-1,n} + f_{m-1,n-1} = f_{m,\ell} + f_{m-1,\ell} + f_{m-1,\ell-1} \qquad (m-1 \geqslant n,\ \ell \geqslant 1).$$

Proof. From (1.3) and (1.4) one gets

(4.2) $$f_{m+1,n} = f_{m,n} + f_{m-1,n} = f_{m,n-1} + f_{m-1,n-2}$$

and

(4.3) $$f_{m,n} + f_{m-1,n} + f_{m-1,n-1} = f_{m,n-1} + f_{m-1,n-2} + f_{m-1,n-1} \cdot$$

This relation is transmitted along a given row m and yields (4.1).

It is easy to derive from (4.1) the following relation

(4.4) $$f_{m,n} + f_{m-1,n} + f_{m-1,n-1} = f_{m+1} \cdot$$

Application of (1.3) to (4.4) gives

(4.5) $$f_{m+1,n} + f_{m-1,n-1} = f_{m+1} \cdot$$

Combination of (4.4) and (4.5) with proper shift of suffices one gets the transmission property pertinent to an upward triangle (Fig. 4b),

(4.6) $$f_{m,n} + f_{m,n-1} - f_{m-1,n-1} = f_m \cdot$$

5. ROLLING DUMBBELL

The relation (4.5) means that the sum of any two vertical neighbors in the Fibonacci Triangle is kept constant for a horizontal movement. Add up the both sides of the two equations derived from (4.5) by substituting $(m = m+2, n = n+1)$ and $(m = m-2, n = n-1)$, subtract (4.5) from the sum, and one gets

(5.1) $$f_{m+3,n+1} + f_{m-3,n-2} = f_{m+2} + f_{m-1} = 2f_{m+1} \cdot$$

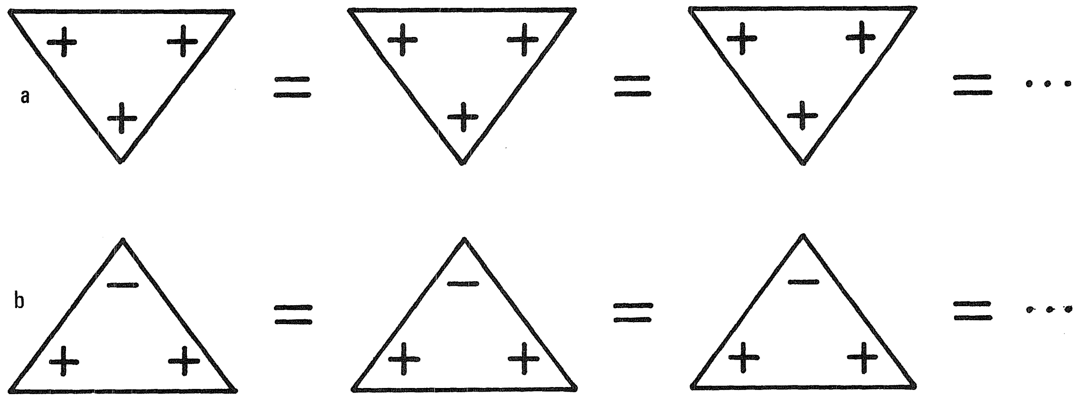

Figure 4. Crawling Crab

More generally one gets the "Rolling Dumbbell" relation (Fig. 5a)

(5.2) $$f_{m+2k+1,n+k} + f_{m-2k-1,n-k-1} = f_{2k} \cdot f_{m+1} = f_{m+2k+1,m+1}$$

(from (2.5)).

By putting $m = m - 2$ and $n = n - 1$ into (4.5) one gets

(5.3) $$f_{m-1,n-1} + f_{m-3,n-2} = f_{m-1}\,.$$

Subtraction of (5.3) from (4.5) followed by substitution $m = m + 1$ and $n = n + 1$ gives

(5.4) $$f_{m+2,n+1} - f_{m-2,n-1} = f_{m+1}\,.$$

This is extended to the expression

(5.5) $$f_{m+2k,n+k} - f_{m-2k,n-k} = f_{2k-1} \cdot f_{m+1} = f_{m+2k,m+1}\,,$$

which is illustrated in Fig. 5b.

6. RELATION WITH THE TOPOLOGICAL INDEX

The present author has defined the topological index Z_G for characterizing the topological nature of a non-directed graph G [2]. A non-adjacent number $p(G,k)$ for graph G is defined as the number of ways in which k disconnected lines are chosen from G; $p(G,0)$ being defined as unity for all the cases. The topological index Z_G is the sum of the $p(G,k)$ numbers.

It is shown [2] that the non-adjacent numbers and the topological index of a path progression S_N* are given by

(6.1) $$p(S_N, k) = \binom{N-k}{k}$$

(6.2) $$Z_{S_N} = \sum_{k=0}^{[N/2]} \binom{N-k}{k} = f_N\,.$$

The topological index of a series of path progressions recurses,

(6.3) $$Z_{S_N} = Z_{S_{N-1}} + Z_{S_{N-2}}\,,$$

as the Fibonacci sequence (1.1). This is a special case of the composition principle (a recursion formula) of Z_G,

(6.4) $$Z_G = Z_{G-\ell} + Z_{G\ominus\ell}$$

*A path progression S_N is a graph composed of linearly connected N points. A point is S_1 and a line joining a pair of given points is S_2 and so on.

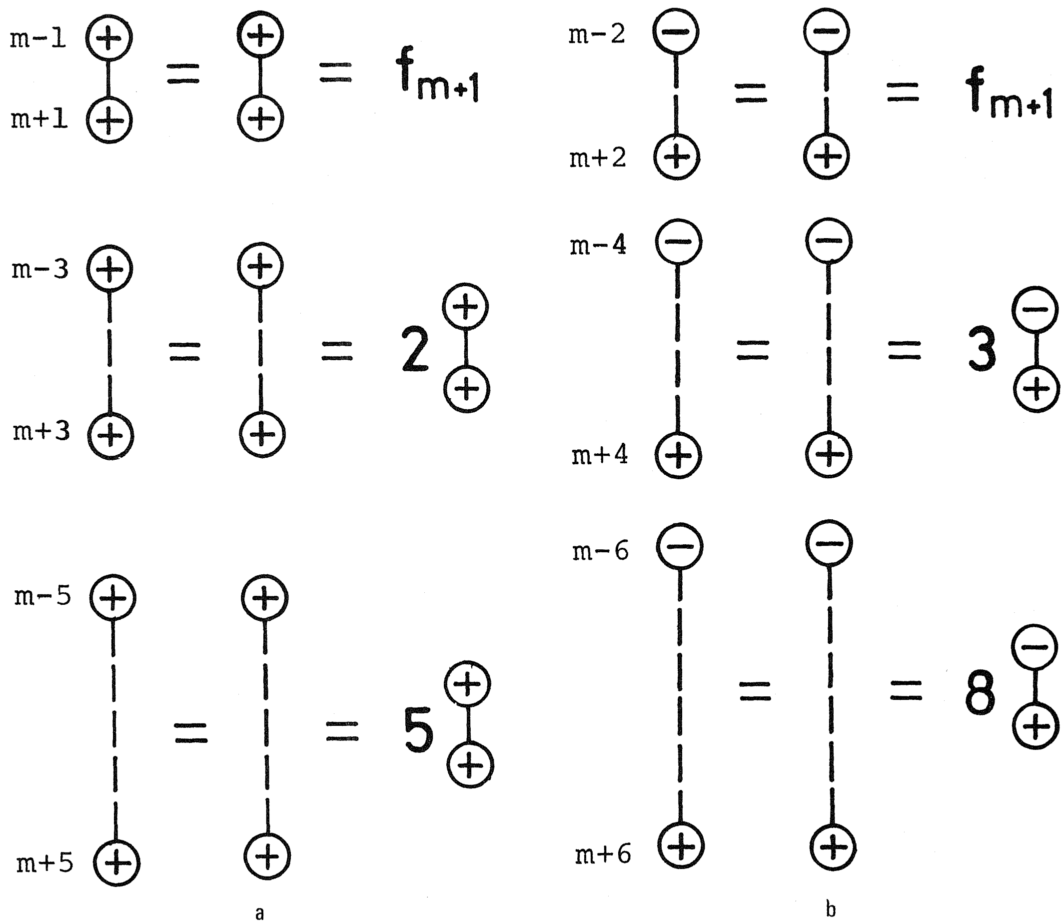

Figure 5. Rolling Dumbbell

where $G - \ell$ is a subgraph of G derived from G by deleting line ℓ, and $G \ominus \ell$ is further derived from $G - \ell$ by deleting all the lines which were adjacent to line ℓ in G.

In Fig. 6 a graphical equivalent of the Fibonacci Triangle is given, where the "roofs" are omitted owing to their redundancy. Note, however, that in this case (1.3)–(1.5) should read

(1.3′) $$f_{m,n} = f_{m-1,n} + f_{m-2,n}$$

(1.4′) $$f_{m,n} = f_{m-1,n-1} + f_{m-2,n-2} \qquad (m \geq 3,\ m-1 \geq n \geq 1)$$

(1.5′) $$f_{1,1} = 1, \qquad f_{2,1} = f_{1,2} = 2, \qquad f_{3,2} = 4.$$

Except for the difference in the boundary conditions all the relations pertinent to the Fibonacci Triangle hold for the topological indices of the trangle array of the graphs in Fig. 6.

REFERENCES

1. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Triangular Numbers," *The Fibonacci Quarterly,* Vol. 12, No. 3 (Oct. 1974), pp. 221–230.
2. H. Hosoya, "Topological Index and Fibonacci Numbers with Relation to Chemistry," *The Fibonacci Quarterly,* Vol. 11, No. 3 (Oct. 1973), pp. 255–266.

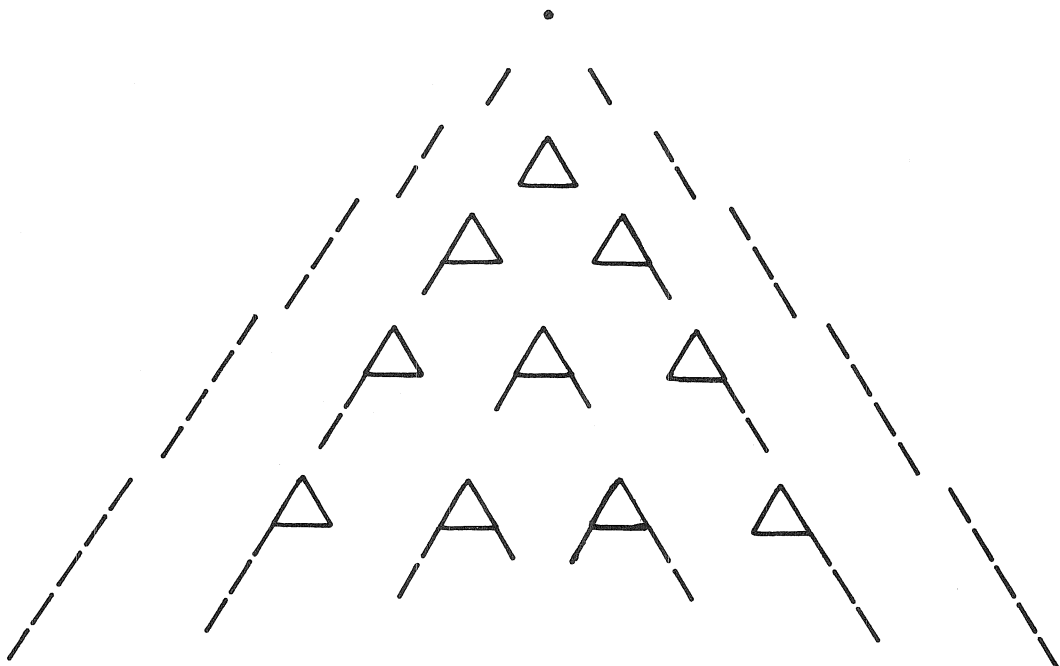

Figure 6. Graphical Equivalent of Fibonacci Triangle

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