ELEMENTARY PROBLEMS AND SOLUTIONS

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-352 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let S_n be defined by $S_0 = 1, S_1 = 2,$ and

$$S_{n+2} = 2S_{n+1} + cS_n.$$

For what value of c is $S_n = 2^n F_{n+1}$ for all nonnegative integers n?

B-353 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

For k and n integers with $0 \leqslant k \leqslant n$, let $A(k,n)$ be defined by $A(0,n) = 1 = A(n,n), A(1,2) = c + 2,$ and

$$A(k+1, n+2) = cA(k,n) + A(k, n+1) + A(k+1, n+1).$$

Also let $S_n = A(0,n) + A(1,n) + \cdots + A(n,n)$. Show that

$$S_{n+2} = 2S_{n+1} + cS_n.$$

B-354 Proposed by Phil Mana, Albuquerque, New Mexico.

Show that

$$F^3_{n+k} - L^3_k F^3_n + (-1)^k F_{n-k}[F^2_{n-k} + 3F_{n+k}F_n L_k] = 0.$$

B-355 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that

$$F^3_{n+k} - L_{3k}F^3_n + (-1)^k F^3_{n-k} = 3(-1)^n F_n F_k F_{2k}.$$

B-356 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let

$$S_n = F_2 + 2F_4 + 3F_6 + \cdots + nF_{2n}.$$

Find m as a function of n so that F_{m+1} is an integral divisor of $F_m + S_n$.

B-357 Proposed by Frank Higgins, Naperville, Illinois.

Let m be a fixed positive integer and let k be a real number such that

$$2m \leqslant \frac{\log(\sqrt{5}k)}{\log a} < 2m + 1,$$

where $a = (1 + \sqrt{5})/2$. For how many positive integers n is $F_n \leqslant k$?

SOLUTIONS

SUM OF SQUARES AS A. P.

B-328 Proposed by Walter Hansell, Mill Valley, California, and V. E. Hoggatt, Jr., San Jose, California

Show that

$$6(1^2+2^2+3^2+\cdots+n^2)$$

is always a sum

$$m^2+(m^2+1)+(m^2+2)+\cdots+(m^2+r)$$

of consecutive integers, of which the first is a perfect square.

Solution by Bob Prielipp, The University of Wisconsin—Oshkosh.

Since

$$6(1^2+2^2+3^2+\cdots+n^2) = n(n+1)(2n+1) = (2n+1)n^2+[2n(2n+1)]/2$$

and

$$m^2+(m^2+1)+(m^2+2)+\cdots+(m^2+r) = (r+1)m^2+[r(r+1)]/2,$$

the desired result follows upon letting $m = n$ and $r = 2n$.

Also solved by Wray G. Brady, Frank Higgins, Mike Hoffman, Herta T. Freitag, Graham Lord, Jeffrey Shallit, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposers.

UNVEILING AN IDENTITY

B-329 Proposed by Herta T. Freitag, Roanoke, Virginia.

Find r, s, and t as linear functions of n such that $2F_r^2 - F_sF_t$ is an integral divisor of $L_{n+2}+L_n$ for $n = 1, 2, \cdots$.

Solution by Mike Hoffman, Warner Robins, Georgia.

Let

$$\alpha = \tfrac{1}{2}(1+\sqrt{5}) \qquad \text{and} \qquad \beta = \tfrac{1}{2}(1-\sqrt{5}).$$

Then

$$\begin{aligned}
2F_r^2 - F_sF_t &= 2\left(\frac{\alpha^r-\beta^r}{\sqrt{5}}\right)^2 - \left(\frac{\alpha^s-\beta^s}{\sqrt{5}}\right)\left(\frac{\alpha^t-\beta^t}{\sqrt{5}}\right)\\
&= 2\,\frac{\alpha^{2r}-2(\alpha\beta)^r+\beta^{2r}}{5} - \frac{\alpha^{s+t}-\alpha^s\beta^t-\beta^s\alpha^t+\beta^{s+t}}{5}\\
&= \frac{2\alpha^{2r}+2\beta^{2r}-\alpha^{s+t}-\beta^{s+t}-4(\alpha\beta)^r+\alpha^s\beta^t+\alpha^t\beta^s}{5}\\
&= \frac{2L_{2r}-L_{s+t}-4(\alpha\beta)^r+(\alpha\beta)^t(\alpha^{s-t}+\beta^{s-t})}{5}\\
&= \frac{2L_{2r}-L_{s+t}+L_{s-t}(-1)^t-4(-1)^r}{5},
\end{aligned}$$

where we have used Binet form for the Fibonacci and Lucas numbers, as well as the fact $\alpha\beta = -1$. Now put $r = n+3$, $s = n+3$, and $t = n-1$. The above becomes

$$\begin{aligned}
2F_r^2 - F_sF_t &= \frac{2L_{2n+2}-L_{2n+1}+L_3(-1)^{n-1}-4(-1)^{n+1}}{5}\\
&= \frac{L_{2n+2}+L_{2n+2}-L_{2n+1}+4(-1)^{n-1}-4(-1)^{n+1}}{5} = \frac{L_{2n+2}+L_{2n}}{5} = F_{2n+1}.
\end{aligned}$$

Thus we have

$$L_{2n+2} + L_{2n} = 5(2F_r^2 - F_sF_t)$$

for positive integers n.

Also solved by the Proposer.

FINDING A G. C. D.

B-330 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Let

$$G_n = F_n + 29F_{n+4} + F_{n+8}.$$

Find the greatest common divisor of the infinite set of integers $\{G_0, G_1, G_2, \cdots\}$.

Solution by Graham Lord, Universite Laval, Quebec, Canada.

It is easy to show that $G_n = 36F_{n+4}$ by using repeatedly the classical Fibonacci recursion relation. Hence, as two consecutive Fibonacci numbers are relatively prime, the g.c.d. of the numbers $G_0, G_1, G_2, \cdots$, is equal to 36.

Also solved by Wray G. Brady, Herta T. Freitag, Frank Higgins, Mike Hoffman, Bob Prielipp, Jeffrey Shallit, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

SOME FIBONACCI SQUARES MOD 24

B-331 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $F_{6n+1}^2 \equiv 1 \pmod{24}$.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

A congruence table of F_n (modulo 24) is

n	1	2	3	4	5	6	7	8	9	10	11	12
F_n (mod 24)	1	1	2	3	5	8	13	21	10	7	17	0

n	13	14	15	16	17	18	19	20	21	22	23	24
F_n (mod 24)	17	17	10	3	13	16	5	21	2	23	1	0

Hence $F_{6n+1} \equiv 1, 13, 17, 5 \pmod{24}$ and $F_{6n+1}^2 \equiv 1 \pmod{24}$.

Also solved by Herta T. Freitag, Frank Higgins, Mike Hoffman, Graham Lord, Bob Prielipp, Sahib Singh, David Zeitlin, and the Proposer.

ONE SINGLE AND ONE TRIPLE PART

B-332 Proposed by Phil Mana, Albuquerque, New Mexico.

Let $a(n)$ be the number of ordered pairs of integers (r,s) with both $0 \leqslant r \leqslant s$ and $2r + s = n$. Find the generating function

$$A(x) = a(0) + xa(1) + x^2a(2) + \cdots.$$

Solution by Graham Lord, Universite Laval, Quebec, Canada.

If s is written as $r + t$, where $t \geqslant 0$ then the decomposition $n = 2r + s$ is the same as $3r + t$, where the only restriction on r and t is that they be nonnegative integers. Thus $a(n)$ counts the number of partitions of n in the form $3r + t$ and so has the generating function

$$A(x) = (1 + x + x^2 + \cdots)\cdot(1 + x^3 + x^6 + x^9 + \cdots) = [(1-x)(1-x)(1-x^3)]^{-1}.$$

Also solved by Wray G. Brady, Frank Higgins, Mike Hoffman, Sahib Singh, Gregory Wulczyn, and the Proposer.

BIJECTION IN $Z^+ \times Z^+$

B-333 Proposed by Phil Mana, Albuquerque, New Mexico.

Let S_n be the set of ordered pairs of integers (a,b) with both $0 < a < b$ and $a + b \leqslant n$. Let T_n be the set of ordered pairs of integers (c,d) with both $0 < c < d < n$ and $c + d > n$. For $n \geqslant 3$, establish at least one bijection (i.e., 1-to-1 correspondence) between S_n and T_{n+1}.

I. Solution by Herta T. Freitag, Roanoke, Virginia; Frank Higgins, Naperville, Illinois; and the Proposer (each separately).

$$c = b \quad \text{and} \quad d = n + 1 - a$$

or inversely,

$$a = n + 1 - d \quad \text{and} \quad b = c.$$

II. Solution by Mike Hoffman, Warner Robins, Georgia; and the Proposer (separately).

$$c = n + 1 - b \quad \text{and} \quad d = n + 1 - a$$

or, inversely,

$$a = n + 1 - d \quad \text{and} \quad b = n + 1 - c.$$

It is straightforward to verify that $a + b \leqslant n$ if and only if $c + d > n$ and hence that each of I and II gives a one-to-one correspondence.

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ADVANCED PROBLEMS AND SOLUTIONS

$$= \frac{x^{\beta+1}w^{-n}}{(1-\beta)x+\beta} \sum_{j=0}^{n} \binom{n}{j} (1 - x^{\beta-1}w)^{-2j} \sum_{m=0}^{\infty} (-1)^{n+j+m} \binom{j}{m} (x^{\beta-1}w)^m (1 + x^{\beta-1}w)^j$$

$$= \frac{x^{\beta+1}(-w)^{-n}}{(1-\beta)x+\beta} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(\frac{1 + x^{\beta-1}w}{1 - x^{\beta-1}w} \right)^j = \frac{x^{\beta+1}(-w)^{-n}}{((1-\beta)x+\beta)} \left(\frac{-2x^{\beta-1}w}{1 - x^{\beta-1}w} \right)^n$$

$$= \frac{x^{\beta+1}2^n}{((1-\beta)x+\beta)} \left(\frac{x^{\beta-1}}{1 - x^{\beta-1}w} \right)^n = \frac{2^n x^{\beta n+\beta+1}}{(1-\beta)x+\beta} .$$

Comparing this with (1), it is clear that we have proved the identity.

CORRECTION

H-267 (Corrected)

Show that

$$S(x) = \sum_{n=0}^{\infty} \frac{(kn+1)^{n-1}X^n}{n!}$$

satisfies $S(x) = e^{xS^k(x)}$.

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