CONVERGENT GENERALIZED FIBONACCI SEQUENCES

WALTER GERDES
Universität Karlsruhe, Federal Republic of Germany

1. INTRODUCTION

In this note we consider sequences of numbers defined by the recursion formula

$$a_{n+2} = \alpha a_{n+1} + \beta a_n, \qquad n = 1, 2, \cdots, \tag{1}$$

with real parameters α, β and arbitrary real numbers a_1, a_2. The sequence $\{a_n\}$ will be called *generalized Fibonacci sequence* and its elements a_n the n^{th} *generalized Fibonacci number*. Sequences like these have been introduced previously by, for example, Bessel-Hagen [1] and Tagiuri [4]. Special cases of (1) are known as the classical Fibonacci sequence with $\alpha = \beta = 1$, $a_1 = a_2 = 1$, the Lucas sequence with $\alpha = \beta = 1$, $a_1 = 1$, $a_2 = 3$, the Pell sequence with $\alpha = 2$, $\beta = 1$, $a_1 = 1$, $a_2 = 2$ and the Fermat sequences with $\alpha = 3$, $\beta = -2$, $a_1 = 1$, $a_2 = 3$ or $a_1 = 2$, $a_2 = 3$. Basic properties of the generalized Fibonacci sequences have been given by A. F. Horadam [3]. In this paper we consider generalized Fibonacci sequences from an analytic point of view. We start with a real representation of the generalized formula of Binet in the second section. In the third section we repeat and complete some properties of finite sums of generalized Fibonacci numbers [3]. With these preparations we are able to characterize convergent generalized Fibonacci sequences in the fourth section and finally in the fifth section we give some limits of *Fibonacci series.*

2. BINET'S FORMULA

For the generalized Fibonacci numbers defined by (1) the (generalized) formula of Binet holds.

$$a_n = \frac{a_2 - a_1 q_2}{q_1 - q_2}\, q_1^{n-1} + \frac{a_1 q_2 - a_2}{q_1 - q_2}\, q_2^{n-1}, \qquad n = 1, 2, \cdots, \tag{2}$$

with q_1, q_2 defined by

$$q_1 = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \beta}, \qquad q_2 = \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + \beta}.$$

The proof of (2) can be given by induction.

Theorem 1. Binet's formula (2) has the following real representations

$$a_n = \begin{cases} \dfrac{a_2 - a_1 q_2}{q_1 - q_2}\, q_1^{n-1} + \dfrac{a_1 q_1 - a_2}{q_1 - q_2}\, q_2^{n-1}, & \dfrac{\alpha^2}{4} + \beta > 0, \quad \text{(3a)} \\[2ex] \left(\dfrac{\alpha}{2}\right)^{n-2} \left[(n-1)a_2 - \dfrac{\alpha}{2}(n-2)a_1 \right], & \dfrac{\alpha^2}{4} + \beta = 0, \quad \text{(3b)} \\[2ex] \dfrac{r^{n-2}}{\sin\phi}\, [a_2 \sin(n-1)\phi - a_1 r \sin(n-2)\phi], & \dfrac{\alpha^2}{4} + \beta < 0, \quad \text{(3c)} \\[2ex] r: = \sqrt{-\beta} > 0, \qquad 0 < \phi: = 2\arctan \dfrac{\sqrt{-\beta} - (\alpha/2)}{\sqrt{-\dfrac{\alpha^2}{4} + \beta}} < \pi, & \end{cases} \quad n = 1, 2, \cdots,$$

with q_1, q_2 defined as in (2).

Proof. Setting $R: = (\alpha^2/4) + \beta$ the case $R > 0$ follows immediately from (2). If $R < 0$, then q_1 and q_2 are the conjugate complex numbers

$$q_{1/2} = \frac{\alpha}{2} \pm i\sqrt{-R}$$

or in polar form $q_{1/2} = re^{\pm i\phi}$ with $r = \sqrt{-\beta} > 0$ and

$$\tan\frac{\phi}{2} = \frac{1-\cos\phi}{\sin\phi} = \frac{\sqrt{-\beta} - \frac{\alpha}{2}}{\sqrt{-R}} \quad \text{or} \quad 0 < \phi = 2\arctan\frac{\sqrt{-\beta} - \frac{\alpha}{2}}{\sqrt{-R}} < \pi,$$

respectively. Further rewriting (2) with $q_2 = \bar{q}_1$

$$a_n = a_1 q_1 \bar{q}_1 \frac{\bar{q}_1^{n-2} - q_1^{n-2}}{q_1 - \bar{q}_1} + a_2 \frac{q_1^{n-1} - \bar{q}_1^{n-1}}{q_1 - \bar{q}_1},$$

employing the polar form mentioned above and using

$$\frac{q_1^m - \bar{q}_1^m}{q_1 - \bar{q}_1} = r^{m-1}\frac{\sin m\phi}{\sin\phi}, \qquad m = 1, 2, \cdots,$$

we conclude statement (3c). (3b) follows from (3c) as limit for $\phi \to 0$. From (3a) and (3b) we get two special cases, which will be useful in the following discussion.

First let $\alpha + \beta = 1$. Then

$$(4) \qquad a_n = \begin{cases} \dfrac{a_2 - a_1}{\alpha - 2}(\alpha - 1)^{n-1} + \dfrac{a_1(\alpha-1) - a_2}{\alpha - 2}, & \alpha \neq 2, \\ (n-1)a_2 - (n-2)a_1, & \alpha = 2, \end{cases} \qquad n = 1, 2, \cdots.$$

Let be $\beta - \alpha = 1$. Then

$$(5) \qquad a_n = \begin{cases} \dfrac{a_2 + a_1}{\alpha + 2}(\alpha + 1)^{n-1} + \dfrac{a_1(\alpha+1) - a_2}{\alpha + 2}(-1)^{n-1}, & \alpha \neq -2, \\ (-1)^n[(n-1)a_2 + (n-2)a_1], & \alpha = -2, \end{cases} \qquad n = 1, 2, \cdots.$$

3. SUMS OF GENERALIZED FIBONACCI NUMBERS

In this chapter we consider some simple properties of finite sums of generalized Fibonacci numbers.

Property 1. The sum of the first n generalized Fibonacci numbers is given by

$$(6a) \qquad \sum_{\nu=1}^{n} a_\nu \frac{1}{\alpha+\beta-1}[a_{n+1} + \beta a_n - a_2 - (1-\alpha)a_1], \quad n = 1, 2, \cdots,$$

if $\alpha + \beta \neq 1$ and by

$$(6b) \qquad \sum_{\nu=1}^{n} a_\nu = \begin{cases} n\dfrac{a_1(\alpha-1) - a_2}{\alpha - 2} + \dfrac{a_1 - a_2}{(\alpha-2)^2}[1 - (\alpha-1)^n], & \alpha \neq 2, \\ \dfrac{n}{2}[n(a_2 - a_1) + 3a_1 - a_2], & \alpha = 2, \end{cases} \quad n = 1, 2, \cdots,$$

if $\alpha + \beta = 1$.

Repeated use of the recursion formula yields statement (6a).

If $\alpha + \beta = 1$, $\alpha \neq 2$, we get the first part of (6b) from (4) using the formula of the finite geometric series. The second part in (6b) follows immediately from (3b) with $\alpha = 2$. Since the following properties can be shown in a similar way, we omit their proofs.

Property 2. The sum of generalized Fibonacci numbers with odd suffixes is given by

$$(7a) \qquad \sum_{\nu=1}^{n} a_{2\nu-1} = a_1 + \frac{1}{\alpha^2 - (\beta-1)^2}[a_{2n} + \beta(1-\beta)a_{2n-1} - \alpha a_2 - \beta(1-\beta)a_1],$$

$n = 1, 2, \cdots$, if $\alpha + \beta \neq 1$, $\beta - \alpha \neq 1$, and by

(7b) $$\sum_{\nu=1}^{n} a_{2\nu-1} = \begin{cases} n\,\dfrac{a_2+a_1(1-\alpha)}{2-\alpha} + \dfrac{a_1-a_2}{\alpha(2-\alpha)^2}\,[1-(\alpha-1)^{2n}], & \alpha \neq 2, \\ n[(n-1)a_2-(n-2)a_1], & \alpha = 2, \end{cases} \quad n = 1, 2, \cdots,$$

if $\alpha+\beta = 1$ and by

(7c) $$\sum_{\nu=1}^{n} a_{2\nu-1} = \begin{cases} n\,\dfrac{a_1(1+\alpha)-a_2}{2+\alpha} - \dfrac{a_1+a_2}{\alpha(2+\alpha)^2}\,[1-(\alpha+1)^{2n}], & \alpha \neq -2, \\ -n[(n-1)a_2+(n-2)a_1], & \alpha = -2, \end{cases} \quad n = 1, 2, \cdots,$$

if $\beta-\alpha = 1$.

Property 3. The sum of generalized Fibonacci numbers with even suffixes is given by

(8a) $$\sum_{\nu=1}^{n} a_{2\nu} = \frac{1}{\alpha^2-(\beta-1)^2}\,[\alpha a_{2n+1}+\beta(1-\beta)a_{2n}+(\beta-1)a_2-\alpha\beta a_1], \quad n = 1, 2, \cdots,$$

if $\alpha+\beta \neq 1$, $\beta-\alpha \neq 1$, and by

(8b) $$\sum_{\nu=1}^{n} a_{2\nu} = \begin{cases} n\,\dfrac{a_2-a_1(\alpha-1)}{2-\alpha} + \dfrac{(\alpha-1)(a_1-a_2)}{\alpha(2-\alpha)^2}\,[1-(\alpha-1)^{2n}], & \alpha \neq 2, \\ n[na_2-(n-1)a_1], & \alpha = 2, \end{cases} \quad n = 1, 2, \cdots,$$

if $\alpha+\beta = 1$ and by

(8c) $$\sum_{\nu=1}^{n} a_{2\nu} = \begin{cases} n\,\dfrac{a_2-a_1(1+\alpha)}{2+\alpha} - \dfrac{(1+\alpha)(a_2+a_1)}{\alpha(2+\alpha)^2}\,[1-(1+\alpha)^{2n}], & \alpha \neq -2, \\ n[na_2+(n-1)a_1], & \alpha = -2, \end{cases} \quad n = 1, 2, \cdots,$$

if $\beta-\alpha = 1$.

Property 4. The sum of generalized Fibonacci numbers with alternating signs is given by

(9a) $$\sum_{\nu=1}^{n} (-1)^{\nu-1}a_\nu = \frac{1}{\alpha-\beta+1}\,[(-1)^{n+1}(a_{n+1}-\beta a_n)-2+(\alpha+1)a_1],$$

$n = 1, 2, \cdots$, if $\beta-\alpha \neq 1$ and by

(9b) $$\sum_{\nu=1}^{n} (-1)^{\nu-1}a_\nu = \begin{cases} n\,\dfrac{a_1(1+\alpha)-a_2}{2+\alpha} + \dfrac{a_1+a_2}{(2+\alpha)^2}\,[1+(-1)^{n-1}(\alpha+1)^n], & \alpha \neq -2, \\ -\dfrac{n}{2}\,[(n-1)a_2+(n-3)a_1], & \alpha = -2, \end{cases}$$

$n = 1, 2, \cdots$, if $\beta-\alpha = 1$.

We terminate this section with one nonlinear property.

Property 5. The sum of squares of the generalized Fibonacci numbers is given by

(10) $$\sum_{\nu=1}^{n} a_\nu^2 = \frac{1}{1+\beta}\,[a_1\sigma_n+(a_2-\alpha a_1)\tau_{n-1}+\beta a_n^2], \quad \beta \neq -1, \quad n = 1, 2, 3, \cdots,$$

with σ_n and τ_n defined by

$$\sigma_n := \sum_{\nu=1}^{n} a_{2\nu-1}, \qquad \tau_n := \sum_{\nu=1}^{n} a_{2\nu}.$$

The explicit form of (10) may be found with the formulas (7) and (8).

4. CONVERGENT FIBONACCI SEQUENCES

Using Binet's formula (2) we are able to characterize the convergent Fibonacci sequences.

Theorem 2. Generalized Fibonacci sequences are convergent if and only if the parameters α, β are points of the region (see Fig. 1)

$$D:=\left\{(\alpha,\beta)\in R^2 \mid \alpha+\beta\leqslant 1,\quad \beta-\alpha<1,\quad \beta>-1\right\}. \tag{11}$$

In the interior $\underline{D}$ of the region D the generalized Fibonacci sequences converge to zero. On the bounday

$$\alpha+\beta=1,\quad 0<\alpha<2,\quad -1<\beta<1,$$

the limit a of the generalized Fibonacci sequences is given by

$$a:=\lim_{n\to\infty} a_n=\frac{a_2+a_1\beta}{1+\beta}. \tag{12}$$

Proof. With the representations (3a)–(3c) for Binet's formula we conclude the following necessary and sufficient conditions for the convergence of the generalized Fibonacci sequences

$$-1<q_1,\ q_2\leqslant 1,\ \frac{\alpha^2}{4}+\beta>0,\ \text{ from (3a)},$$

$$\left|\frac{\alpha}{2}\right|<1,\ \frac{\alpha^2}{4}+\beta=0,\ \text{ from (3b)},$$

$$r=\sqrt{-\beta}<1,\ \frac{\alpha^2}{4}+\beta<0,\ \text{ from (3c)}.$$

This means in detail in (3a)

$$-1<\frac{\alpha}{2}+\sqrt{\frac{\alpha^2}{4}+\beta}\leqslant 1,$$

which leads together with $\frac{\alpha^2}{4}+\beta>0$ to $\alpha+\beta\leqslant 1$, $\alpha<2$, and in an analogous way from

$$-1<\frac{\alpha}{2}-\sqrt{\frac{\alpha^2}{4}+\beta}\leqslant 1$$

to $\beta-\alpha<1$, $\alpha>-2$, by (3b) we have $\alpha^2=-4\beta$, $-2<\alpha<2$, and from (3c) $\alpha^2<-4\beta$, $\beta>-1$. All these conditions yield the required convergence domain D for the parameters α, β. In $\underline{D}$ it follows from

$$\lim_{n\to\infty} q_{1/2}^n=0,\ |q_{1/2}|<1,\quad \text{from } \lim_{n\to\infty} n\left(\frac{\alpha}{2}\right)^n=0,\ \left|\frac{\alpha}{2}\right|<1,$$

and from

$$\lim_{n\to\infty} r^n=0,\ 0<r<1,$$

that all limits vanish. On the boundary of D, $\alpha+\beta=1$, $|\alpha|<2$, we get from (4) for $n\to\infty$ the required result[1]

$$a:=\lim_{n\to\infty} a_n=\frac{a_1(\alpha-1)-a_2}{\alpha-2}=\frac{a_2+\beta a_1}{1+\beta}.$$

5. FIBONACCI SERIES

Finally we will consider some Fibonacci series, which are defined as convergent series with generalized Fibonacci numbers as terms. Since terms of convergent series necessarily converge to zero, we have to choose the parameters α, β from the interior $\underline{D}$ of the convergence domain D (11). Tending n to infinity and using Theorem 2 we get the following limits from the properties 1–5:

$$\sum_{\nu=1}^{\infty} a_\nu=\frac{a_2+(1-\alpha)a_1}{1-\alpha-\beta},\qquad (\alpha,\beta)\in\underline{D}, \tag{13}$$

[1] In [2] a special case of this general result is mentioned with $\alpha=\beta=½$, $a=(a_1+2a_2)/3$. This result is obtained by the only use of the recurrence relation (1).

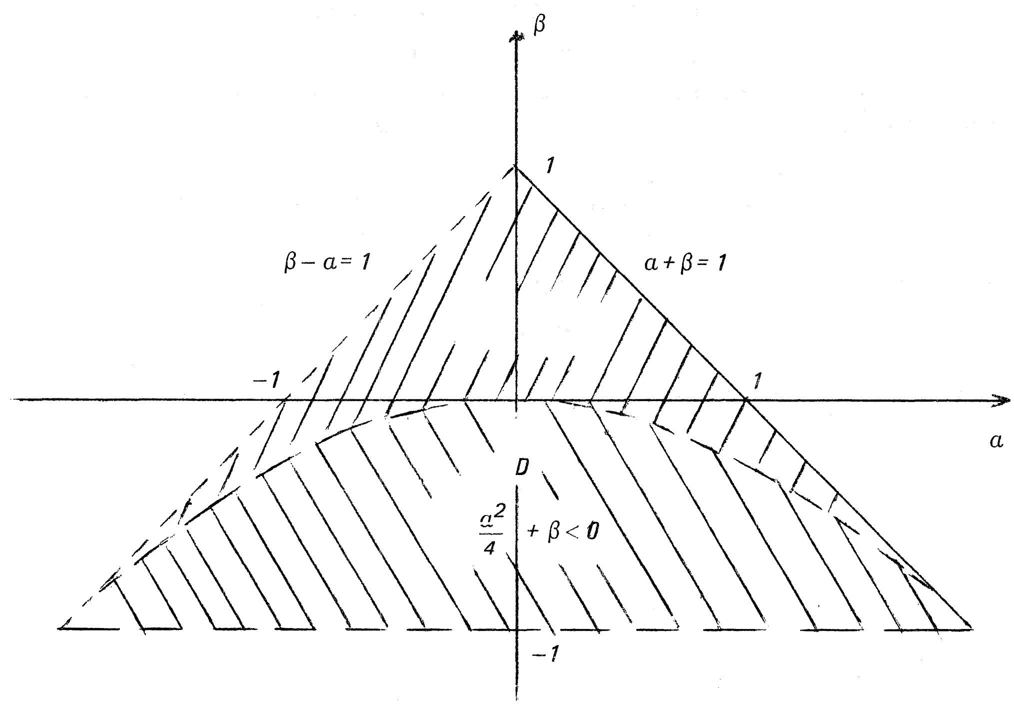

Fig. 1 Region $\underline{D}$ of Convergence of Generalized Fibonacci Sequences

(14) $$\sum_{\nu=1}^{\infty} a_{2\nu-1} = \frac{(\alpha^2+\beta-1)a_1 - \alpha a_2}{\alpha^2-(\beta-1)^2}, \qquad (\alpha,\beta) \in \underline{D},$$

(15) $$\sum_{\nu=1}^{\infty} a_{2\nu} = \frac{(\beta-1)a_2 - \alpha\beta a_1}{\alpha^2-(\beta-1)^2}, \qquad (\alpha,\beta) \in \underline{D},$$

(16) $$\sum_{\nu=1}^{\infty} (-1)^{\nu-1} a_\nu = \frac{(\alpha+1)a_1 - a_2}{\alpha-\beta+1}, \qquad (\alpha,\beta) \in \underline{D},$$

(17) $$\sum_{\nu=1}^{\infty} a_\nu^2 = \frac{a_2^2(\beta-1) - 2\alpha\beta a_1 a_2 + [\alpha^2(1+\beta)+\beta-1]a_1^2}{(1+\beta)[\alpha^2-(\beta-1)^2]}, \qquad (\alpha,\beta) \in \underline{D}.$$

Naturally this list can be extended to other, e.g., cubic or binomial, sums using Theorem 2.

REFERENCES

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