ON GENERATING FUNCTIONS WITH COMPOSITE COEFFICIENTS

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In this paper, we shall investigate a general problem of an interesting nature, and indicate a systematic method for obtaining at least a partial solution of it. The problem we mean is this: given the three generating functions (assuming appropriate convergence limitations are satisfied):

$$(1)\qquad f(u) = \sum_{n=0}^{\infty} a_n u^n; \qquad g(u) = \sum_{n=0}^{\infty} b_n u^n; \qquad h(u) = \sum_{n=0}^{\infty} a_n b_n u^n .$$

What is the relationship, if any, which exists between $f(u)$, $g(u)$ and $h(u)$? By "relationship," we shall here mean that $h(u)$ may be expressed explicitly and in closed form as a function of u.

Many such relationships are well known, a few of which are indicated below, in tabular form:

a_n	$f(u)$	b_n	$g(u)$	$a_n b_n$	$h(u)$
p^n	$(1-pu)^{-1}$	q^n	$(1-qu)^{-1}$	$p^n q^n$	$(1-pqu)^{-1}$
$(-1)^n \binom{x}{n}$	$(1-u)^x$	q^n	$(1-qu)^{-1}$	$(-q)^n \binom{x}{n}$	$(1-qu)^x$
$\frac{1}{n!}$	e^u	$\frac{(-1)^n}{n!}$	e^{-u}	$\frac{(-1)^n}{n!n!}$	$J_0(2\sqrt{u})$
$\frac{(-1)^n}{n+1}$	$\frac{1}{u}\ln(1+u)$	$\frac{(-1)^n}{n+1}$	$\frac{1}{u}\ln(1+u)$	$\frac{1}{(n+1)^2}$	$\frac{-1}{u}\int_0^u \frac{\ln(1-t)}{t}\,dt$

As the last example illustrates, our general problem encompasses that of determining $h(u)$ when $f(u) = g(u)$, i.e., when

$$h(u) = \sum_{n=0}^{\infty} a_n^2 u^n ;$$

this latter, more specific case, is discussed briefly by Gould [2]. Our approach to the problem will depend on finite difference methods.

We recall the unit difference operators E and Δ, satisfying the following formal relationships (assuming arbitrary operand θ_o):

$$(2)\qquad E^n\theta_0 = (1+\Delta)^n\theta_0 = \theta_n$$

$$(3)\qquad \Delta^n\theta_o = (E-1)^n\theta_0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E^k\theta_0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\theta_k .$$

Using the above relationships, we may develop $h(u)$ as follows:

$$h(u) = \sum_{n=0}^{\infty} a_n u^n E^n b_0 = \sum_{n=0}^{\infty} a_n (uE)^n b_0 = f(uEb_0),$$

or

$$(4)\qquad h(u) = f(u+u\Delta_b) = g(u+u\Delta_a) \quad \text{(the latter by symmetry).}$$

Naturally, we are taking great liberties in treating the formal operators E and Δ as if they were algebraic quantities, not to mention the fact that we are also ignoring convergence restrictions, if any. However, these objections may be circumvented if we treat the functions in (1) as formal generating functions, and focus our attention on their coefficients. We shall demonstrate that if due care is exercised in the manipulation of the operators and their operands, relation (4) may be made to yield results which are consistent with known relationships. In the process, we will also obtain some interesting and sometimes useful identities as by-products. Without further ado, we will illustrate the applicability of (4), first in obtaining the results already tabulated, then in developing other, more general relationships.

EXAMPLE 1. We begin by applying (3) to

$$a_o : \Delta^n a_0 = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}p^k = (p-1)^n .$$

Then, using our result in (4),

$$h(u) = g(u+u\Delta_a) = (1-qu-qu\Delta_a)^{-1} = (1-qu)^{-1}\left(1-\frac{qu\Delta_a}{1-qu}\right)^{-1} = (1-qu)^{-1}\sum_{n=0}^{\infty}\left(\frac{qu}{1-qu}\right)^n \Delta^n a_0$$

$$= (1-qu)^{-1}\sum_{n=0}^{\infty}\left(\frac{qu}{1-qu}\right)^n (p-1)^n = (1-qu)^{-1}\left(1-\frac{(p-1)qu}{1-qu}\right)^{-1} = (1-qu-(p-1)qu)^{-1}$$

$$= (1-pqu)^{-1},$$

as was previously stated. We could just as easily have used the relation in (4) obtained by reversing the roles of $f(u)$ and $g(u)$, and of a_n and b_n. The end result would have been identical.

EXAMPLE 2. By formula (3),

$$\Delta^n a_0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(-1)^k\binom{x}{k} = (-1)^n\sum_{k=0}^{n}\binom{x}{n}\binom{n}{n-k} = (-1)^n\binom{x+n}{n} = \binom{-x-1}{n}$$

(using, e.g., formula (3.1) in [1]). As in Example 1,

$$h(u) = g(u+u\Delta_a) = (1-qu)^{-1}\sum_{n=0}^{\infty}\left(\frac{qu}{1-qu}\right)^n\Delta^n a_0 = (1-qu)^{-1}\sum_{n=0}^{\infty}\binom{-x-1}{n}\left(\frac{qu}{1-qu}\right)^n$$

$$= (1-qu)^{-1}\left(1+\frac{qu}{1-qu}\right)^{-x-1} = (1-qu)^x ,$$

as stated. It is instructive to reverse the roles of $f(u)$ and $g(u)$ in this example:

$$h(u) = f(u+u\Delta_b) = (1-u-u\Delta_b)^x = (1-u)^x\left(1-\frac{u\Delta_b}{1-u}\right)^x = (1-u)^x\sum_{n=0}^{\infty}\binom{x}{n}\left(\frac{-u}{1-u}\right)^n\Delta^n b_0$$

$$= (1-u)^x\sum_{n=0}^{\infty}\binom{x}{n}\left(\frac{-u}{1-u}\right)^n(q-1)^n, \quad \text{(Using Example 1)}$$

$$= (1-u)^x\left(1-\frac{(q-1)u}{1-u}\right)^x = (1-u-(q-1)u)^x = (1-qu)^x, \quad \text{as before.}$$

EXAMPLE 3.

$$\Delta^n a_0 = \sum_{k=0}^{n}(-1)^{n-k}\frac{\binom{n}{k}}{k!},$$

using (3). This may be expressed in terms of an ordinary Laguerre polynomial as $(-1)^n L_n(1)$ (see fomula 1.115 in [1]); however, we will leave it in the summation form, to demonstrate that it is not essential for the n^{th} difference of the coefficients to be represented in closed form.

Then,

$$h(u) = e^{-u-u\Delta_a} = e^{-u}\sum_{n=0}^{\infty}\frac{(-u)^n}{n!}\Delta^n a_0 = e^{-u}\sum_{n=0}^{\infty}\frac{(-u)^n}{n!}\sum_{k=0}^{n}(-1)^{n-k}\frac{\binom{n}{k}}{k!}$$

$$= e^{-u}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{n=k}^{\infty}\binom{n}{k}\frac{u^n}{n!} = e^{-u}\sum_{k=0}^{\infty}\frac{(-u)^k}{k!\,k!}\sum_{n=0}^{\infty}\frac{u^n}{n!} = e^{-u}\sum_{k=0}^{\infty}\frac{(-u)^k}{k!\,k!}e^u = \sum_{k=0}^{\infty}\frac{(-u)^k}{k!\,k!}$$

$$= J_0(2\sqrt{u}),$$

by definition of the Bessel function.

EXAMPLE 4.

$$\Delta^n a_0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{(-1)^k}{k+1} = (-1)^n\sum_{k=0}^{n}\frac{\binom{n}{k}}{k+1}.$$

We may use formula (1.37) in [1] to evaluate this expression (for the case $x = 1$), and find that

$$\Delta^n a_0 = \frac{(-1)^n}{n+1}(2^{n+1}-1).$$

Now

$$h(u) = g(u(1+\Delta_a)) = \frac{\ln(1+(1+\Delta_a)u)}{u(1+\Delta_a)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}u^n(1+\Delta_a)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}u^n\sum_{k=0}^{n}\binom{n}{k}\Delta^k a_0$$

$$= \sum_{k=0}^{\infty}\Delta^k a_0\sum_{n=k}^{\infty}\frac{(-1)^n}{n+1}\binom{n}{k}u^n = \sum_{k=0}^{\infty}\Delta^k a_0(-1)^k\sum_{n=0}^{\infty}\frac{(-1)^n\binom{n+k}{n}}{n+k+1}u^{n+k}.$$

Denote the latter expression by y; if we differentiate uy with respect to u, we obtain:

$$(uy)' = \sum_{k=0}^{\infty}(-1)^k\Delta^k a_0\sum_{n=0}^{\infty}(-1)^n\binom{n+k}{n}u^{n+k} = \sum_{k=0}^{\infty}(-1)^k\frac{(-1)^k}{k+1}(2^{k+1}-1)u^k(1+u)^{-k-1}$$

$$= \frac{2}{u+1}\sum_{k=0}^{\infty}\frac{\left(\frac{2u}{1+u}\right)^k}{k+1} - \frac{1}{1+u}\sum_{k=0}^{\infty}\frac{\left(\frac{u}{1+u}\right)^k}{k+1} = \frac{-2(1+u)}{(1+u)(2u)}\ln\left(1-\frac{2u}{1+u}\right) + \frac{1+u}{u(1+u)}\ln\left(1-\frac{u}{1+u}\right)$$

$$= -\frac{1}{u}\ln\left(\frac{1-u}{1+u}\right) - \frac{1}{u}\ln(1+u) = -\frac{1}{u}\ln(1-u).$$

We may now integrate with respect to u, noting that $uy = 0$ when $u = 0$, and we arrive at the desired expression:

$$uy = uh(u) = \int_0^u \frac{-1}{t}\ln(1-t)dt, \quad \text{or} \quad h(u) = \frac{-1}{u}\int_0^u \frac{\ln(1-t)}{t}dt.$$

We should observe that, in the foregoing examples, we could have arrived at the desired closed form of $h(u)$ by more direct methods, and thus we have not really saved ourselves any effort. On the other hand, these particular examples were chosen precisely for their simplicity, so as to enable us to check on the consistency of our results. In what follows, the value of our method will become apparent. Also, in the examples, both $f(u)$ and $g(u)$ were given in closed form (in fact, in terms of elementary functions). It is required that only one of these two functions be given in closed form; then, the n^{th} difference will be taken on the coefficients of the other function. If both $f(u)$ and $g(u)$ are given in closed form, however, as we have seen, we may develop $h(u)$ in either of two ways, which should yield the same result.

We shall begin by "proving" a well known linear transformation for the Gaussian hypergeometric function, defined as follows:

(5) $$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{\binom{-a}{n}\binom{-b}{n}}{\binom{-c}{n}} (-z)^n .$$

Let

$$f(u) = \sum_{n=0}^{n} (-1)^n \binom{x}{n} u^n = (1-u)^x, \quad \text{and} \quad g(u) = \sum_{n=0}^{\infty} (-1)^n \binom{y}{n} u^n = (1-u)^y .$$

From (5), we see that

$$h(u) = \sum_{n=0}^{\infty} \binom{x}{n}\binom{y}{n} u^n = F(-x,-y,1;u).$$

But using (4), $h(u)$ may be expressed in the alternative form

$$(1-u-u\Delta_a)^y = (1-u)^y\left(1-\frac{u\Delta_a}{1-u}\right)^y = (1-u)^y \sum_{n=0}^{\infty} \binom{y}{n} \Delta_a^n \left(\frac{-u}{1-u}\right)^n$$

$$= (1-u)^y \sum_{n=0}^{\infty} \binom{y}{n}\binom{-x-1}{n}\left(\frac{-u}{1-u}\right)^n \quad \text{(using Example 1)},$$

$$= (1-u)^y F\left(-y, x+1, 1; \frac{-u}{1-u}\right) = (1-u)^x F\left(-x, y+1, 1; \frac{-u}{1-u}\right) \quad \text{(by symmetry)}.$$

These last relations may be found in [3], as formulas 15.3.4 and 15.3.5, setting $a=-x$, $b=-y$, $c=1$ and $z=u$. As an interesting special case, if we set $y=-x-1$, and $-u/(1-u)=w$, we obtain the following:

(6) $$(1-w)^{x+1} F(x+1, x+1, 1; w) = (1-w)^{-x} F(-x, -x, 1; w)$$

This is equivalent to formula (3.141) in [1].

Other important special cases of the hypergeometric linear transformation given above occur whenever either x or y are positive integers, causing the series to terminate after a finite number of terms. For example, if $x=3$ and $y=2$, our relation yields:

$$F(-3,-2,1;u) = (1-u)^2 F\left(-2,4,1;\frac{-u}{1-u}\right) = (1-u)^3 F\left(-3,3,1;\frac{-u}{1-u}\right),$$

each expression reducing to $1+6u+3u^2$.

Another set of interesting special cases is obtained by setting $x=y=-m-1$, where m is a non-negative integer. This yields the identity:

$$\sum_{n=0}^{\infty} \binom{m+n}{n}^2 u^n = (1-u)^{-m-1} \sum_{n=0}^{m} \binom{m}{n}\binom{m+n}{n}\left(\frac{u}{1-u}\right)^n .$$

If we obtain the convolute of (6), we obtain the identity:

(7) $$\sum_{k=0}^{n} \binom{-x-1}{k}^2 \binom{x+1}{n-k}(-1)^k = \sum_{k=0}^{n} \binom{x}{k}^2 \binom{-x}{n-k}(-1)^k .$$

A more general identity is obtained by expanding each side of the general linear transformation formula, after making the substitution $-u/(1-u)=w$:

(8) $$\sum_{k=0}^{n} (-1)^k \binom{-x-1}{k}\binom{y}{k}\binom{-y}{n-k} = \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{-y-1}{k}\binom{-x}{n-k} .$$

For our second application of (4), we shall use $f(u) = e^u$, and

$$b_n = \frac{\binom{x}{n}}{\binom{y}{n}} \qquad ;$$

then $a_n = 1/n!$, and $g(u) = F(-x, 1, -y; u)$. Using identity (3),

$$\Delta^n b_0 = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k} \frac{\binom{x}{k}}{\binom{y}{k}} = (-1)^n \frac{\binom{y-x}{n}}{\binom{y}{n}}$$

(equivalent to (7.1) in [1]). Now

$$h(u) = e^{u+u\Delta_b} = e^u \sum_{n=0}^{\infty} \frac{u^n}{n!}\Delta^n b_0 = e^u \sum_{n=0}^{\infty} \frac{(-u)^n \binom{y-x}{n}}{\binom{y}{n} n!} = e^u M(x-y, -y, -u),$$

where $M(a,b,z)$ is the confluent hypergeometric function, or Kummer function, defined as follows:

$$\text{(9)} \qquad M(a,b,z) = \sum_{n=0}^{\infty} \frac{\binom{-a}{n}}{\binom{-b}{n}} \frac{z^n}{n!}$$

(see, e.g., pp. 504–505 of [3]). Since $h(u)$ is also equal to

$$\sum_{n=0}^{\infty} \frac{\binom{x}{n}}{\binom{y}{n}} \frac{u^n}{n!} = M(-x, -y, u),$$

we have derived the basic transformation formula for the Kummer function:

$$M(x, y, u) = e^u M(y-x, y, -u),$$

substituting $-x$ and $-y$ for x and y, respectively.

As another application, we will prove the following identity:

$$\text{(10)} \qquad f(r,u) = \sum_{n=0}^{\infty} \left\{\frac{p^n - q^n}{p-q}\right\}^r u^n = (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{1 - p^{r-k}q^k u}$$

The proof is by induction. We denote the assertion that (10) is true for a non-negative integer r as $P(r)$. We observe that $P(0)$ implies that $f(0,u) = (1-u)^{-1}$, while $P(1)$ implies that

$$f(1,u) = \frac{(1-pu)^{-1} - (1-qu)^{-1}}{p-q} \qquad ,$$

each assertion readily verifiable as being true. We assume the validity of $P(r)$. Also, we define $g(u) = f(1,u)$ and $h(u) = f(r+1, u)$, consequently. By application of the result found in Example 1, since

$$b_n = \frac{p^n - q^n}{p-q}, \qquad \Delta^n b_0 = \frac{(p-1)^n - (q-1)^n}{p-q} .$$

Also,

$$h(u) = f(r, u + u\Delta_b) = (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{1 - p^{r-k}q^k u - p^{r-k}q^k u\Delta_b} = (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{(1-p^{r-k}q^k u)\left(1 - \dfrac{p^{r-k}q^k u\Delta_b}{1-p^{r-k}q^k u}\right)}$$

$$= (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{1 - p^{r-k}q^k u} \sum_{i=0}^{\infty} \left\{\frac{p^{r-k}q^k u}{1-p^{r-k}q^k u}\right\}^i \Delta^i b_0$$

$$= (p-q)^{-r} \sum_{k=0}^{r} \frac{(-1)^k \binom{r}{k}}{1 - p^{r-k}q^k u} \sum_{i=0}^{\infty} \left\{\frac{p^{r-k}q^k u}{1-p^{r-k}q^k u}\right\}^i \left\{\frac{(p-1)^i - (q-1)^i}{p-q}\right\} =$$

$$= (p-q)^{-r-1}\sum_{k=0}^{r}\frac{(-1)^k\binom{r}{k}}{1-p^{r-k}q^ku}\left\{\left(1-\frac{(p-1)p^{r-k}q^ku}{1-p^{r-k}q^ku}\right)^{-1}-\left(1-\frac{(q-1)p^{r-k}q^ku}{1-p^{r-k}q^ku}\right)^{-1}\right\}$$

$$= (p-q)^{-r-1}\sum_{k=0}^{r}(-1)^k\binom{r}{k}\left\{(1-p^{r+1-k}q^ku)^{-1}-(1-p^{r-k}q^{k+1}u)^{-1}\right\}$$

$$= (p-q)^{-r-1}\left\{\sum_{k=0}^{r}\frac{(-1)^k\binom{r}{k}}{1-p^{r+1-k}q^ku}-\sum_{k=1}^{r+1}\frac{(-1)^{k-1}\binom{r}{k-1}}{1-p^{r+1-k}q^ku}\right\}$$

$$= (p-q)^{-r-1}\left\{\frac{1}{1-p^{r+1}u}+\sum_{k=1}^{r}\frac{(-1)^k\left\{\binom{r}{k}+\binom{r}{k-1}\right\}}{1-p^{r+1-k}q^ku}-\frac{(-1)^r}{1-q^{r+1}u}\right\}$$

$$= (p-q)^{-r-1}\left\{\frac{1}{1-p^{r+1}u}+\sum_{k=1}^{r}\frac{(-1)^k\binom{r+1}{k}}{1-p^{r+1-k}q^ku}+\frac{(-1)^{r+1}}{1-q^{r+1}u}\right\}$$

or

$$f(r+1,u) = (p-q)^{-r-1}\sum_{k=0}^{r+1}\frac{(-1)^k\binom{r+1}{k}}{1-p^{r+1-k}q^ku},$$

which equals $P(r+1)$. Therefore, $P(r)\to P(r+1)$, completing the proof.

If we set $p=\alpha$ and $q=\beta$ in (10) (the familiar Fibonacci constants), we obtain the generating function for the r^{th} power of the Fibonacci numbers, in the form of a partial fraction series:

(10a)
$$\sum_{n=0}^{\infty}F_n^ru^n = 5^{-\frac{1}{2}r}\sum_{k=0}^{r}\frac{(-1)^k\binom{r}{k}}{1-\alpha^{r-k}\beta^ku}.$$

By a very similar development, we can prove the following identity:

(11)
$$\sum_{n=0}^{\infty}\left\{\frac{p^n+q^n}{p+q}\right\}^ru^n = (p+q)^{-r}\sum_{k=0}^{r}\frac{\binom{r}{k}}{1-p^{r-k}q^ku}.$$

Again, with $p=\alpha$ and $q=\beta$ in (11), we obtain the generating function for the r^{th} power of the Lucas numbers:

(11a)
$$\sum_{n=0}^{\infty}L_n^ru^n = \sum_{k=0}^{r}\frac{\binom{r}{k}}{1-\alpha^{r-k}\beta^ku}.$$

We may combine the partial fractions in (10a) and (11a), using known Fibonacci and Lucas identities, to eliminate all irrational expressions and condense the result in one closed form. For example, if $A(r,u)$ and $B(r,u)$ denote the expressions in (10a) and (11a), respectively, we may obtain the following results:

$$A(1,u) = u/(1-u-u^2);\quad A(2,u) = \frac{u-u^2}{1-2u-2u^2+u^3};$$

$$A(3,u) = (u-2u^2-u^3)/(1-3u-6u^2+3u^3+u^4);\quad B(1,u) = (2-u)/(1-u-u^2);$$

$$B(2,u) = (4-7u-u^2)/(1-2u-2u^2+u^3);\quad B(4,u) = \frac{16-79u-164u^2+76u^3+u^4}{1-5u-15u^2+15u^3+5u^4-u^5},$$

etc.

The possibilities for applying our method are virtually unlimited, provided we are careful not to separate the Δ operator in its manipulations. In this respect, Δ does not behave like an ordinary algebraic quantity, since "multiplication" is really successive application of the Δ symbol. Except for very special cases, moreover, which must be treated separately, a closed form for $h(u)$ free of symbolic operators is generally not available. The readers are invited to find other examples where the indicated method can yield useful results.

In a forthcoming paper on the topic (viz. [4]), an alternative (and more rigorous) approach is presented for the general solution of the problem proposed in this paper, under appropriate restrictions of analyticity for functions f and g.

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[Continued from p. 268.]

$$T_{i-3} = \sum_{m=0}^{\left[\frac{i-3}{2}\right]} \sum_{r=0}^{\left[\frac{i-3}{2}\right]} \binom{i-m-2r-3}{m+r}\binom{m+r}{r} = \sum_{m=2}^{\left[\frac{i+1}{2}\right]} \sum_{r=1}^{\left[\frac{i-1}{3}\right]} \binom{i-m-2r-1}{m+r-1}\binom{m+r-1}{r-1} .$$

Now,

$$T_i = T_{i-1} + T_{i-2} + T_{i-3} = \sum_{m=0}^{[i/2]} \sum_{r=0}^{[i/3]} \binom{i-m-2r}{m+r}\binom{m+r}{r}$$

(from lemma) which is what we required.

Fairly clearly when we are in the plane $r = 0$, we have the ordinary Fibonacci numbers. Further investigations suggest themselves along the lines of Hoggatt [3] and Horner [4].

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