TRIBONACCI NUMBERS AND PASCAL'S PYRAMID

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In this paper an expression for the Tribonacci numbers discussed by the late Mark Feinberg [1] is obtained. They are expressed as sums of numbers along diagonal planes of what might be called Pascal's pyramid.

Feinberg [2] used the coefficients of a trinomial expansion as the model of a three-dimensional pyramid. He projected this pyramid onto a plane and then added the diagonal lines to get $\{T_n\}$, the Tribonacci sequence, $\{1, 1, 2, 4, 7, 13, 24, 44, \cdots\}$.

Lemma.

$$\binom{n-m-2r}{m+r}\binom{m+r}{r} = \binom{n-m-2r-1}{m+r}\binom{m+r}{r} + \binom{n-m-2r-1}{m+r-1}\binom{m+r-1}{r} + \binom{n-m-2r-1}{m+r-1}\binom{m+r-1}{r-1}.$$

Proof. The last two terms on the right-hand side

$$= \binom{n-m-2r-1}{m+r-1}\left\{\binom{m+r-1}{r} + \binom{m+r-1}{r-1}\right\} = \binom{n-m-2r-1}{m+r-1}\binom{m+r}{r}.$$

$$\binom{n-m-2r-1}{m+r-1}\binom{m+r}{r} + \binom{n-m-2r-1}{m+r}\binom{m+r}{r} = \binom{n-m-2r}{m+r}\binom{m+r}{r},$$

as required.

Theorem.

$$T_n = \sum_{m=0}^{[n/2]}\sum_{r=0}^{[n/3]}\binom{n-m-2r}{m+r}\binom{m+r}{r}.$$

Proof. We use induction.

$$T_0 = T_1 = 1.$$

$$T_2 = \sum_{m=0}^{1}\binom{2-m}{m} = \binom{2}{0} + \binom{1}{1} = 2.$$

$$T_3 = \sum_{m=0}^{1}\sum_{r=0}^{1}\binom{3-m-2r}{m+r}\binom{m+r}{r} = \binom{3}{0} + \binom{2}{1} + \binom{1}{1}\binom{1}{1} + \binom{0}{2}\binom{2}{1} = 4.$$

Assume true for $n = 4, 5, 6, \cdots, i-1$.

$$T_{i-1} = \sum_{m=0}^{\left[\frac{i-1}{2}\right]}\sum_{r=0}^{\left[\frac{i-1}{3}\right]}\binom{i-m-2r-1}{m+r}\binom{m+r}{r}.$$

$$T_{i-2} = \sum_{m=0}^{\left[\frac{i-2}{2}\right]}\sum_{r=0}^{\left[\frac{i-2}{2}\right]}\binom{i-m-2r-2}{m+r}\binom{m+r}{r} = \sum_{m=1}^{\left[\frac{i}{2}\right]}\sum_{r=0}^{\left[\frac{i-2}{2}\right]}\binom{i-m-2r-1}{m+r-1}\binom{m+r-1}{r}.$$

[Continued on p. 275.]