ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-364 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Find and prove a formula for the number $R(n)$ of positive integers less than 2^n whose base 2 representations contain no consecutive 0's. (Here n is a positive integer.)

B-365 Proposed by Phil Mana, Albuquerque, New Mexico.

Show that there is a unique integer $m > 1$ for which integers a and r exist with $L_n \equiv ar^n \pmod{m}$ for all integers $n \geqslant 0$. Also show that no such m exists for the Fibonacci numbers.

B-366 Proposed by Wray G. Brady, University of Tennessee, Knoxville, Tennessee, and Slippery Rock State College, Slippery Rock, Pennsylvania

Prove that $L_i L_j \equiv L_h L_k \pmod{5}$ when $i + j = h + k$.

B-367 Proposed by Gerald E. Bergum, So. Dakota State University, Brookings, So. Dakota.

Let $[x]$ be the greatest integer in x, $a = (1+\sqrt{5})/2$, and $n \geqslant 1$. Prove that

(a) $$F_{2n} = [aF_{2n-1}],$$

and

(b) $$F_{2n+1} = [a^2 F_{2n-1}].$$

B-368 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain functions $g(n)$ and $h(n)$ such that

$$\sum_{i=1}^{n} iF_i L_{n-i} = g(n)F_n + h(n)L_n$$

and use the results to obtain congruences modulo 5 and 10.

B-369 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

For all integers $n \geqslant 0$, prove that the set

$$S_n = \{L_{2n+1}, L_{2n+3}, L_{2n+5}\}$$

has the property that if $x, y \in S_n$ and $x \neq y$ then $xy + 5$ is a perfect square. For $n = 0$ verify that there is no integer z that is not in S_n and for which $\{z, L_{2n+1}, L_{2n+3}, L_{2n+5}\}$ has this property. (For $n > 0$ the problem is unsolved.)

SOLUTIONS

BICENTENNIAL SEQUENCE

B-340 Proposed by Phil Mana, Albuquerque, New Mexico.

Characterize a sequence whose first 28 terms are:

1779, 1784, 1790, 1802, 1813, 1819, 1824, 1830, 1841, 1847, 1852, 1858, 1869, 1875, 1880, 1886, 1897, 1909, 1915, 1920, 1926, 1937, 1943, 1948, 1954, 1965, 1971, 1976.

I. Solution by H. Turner Laquer, University of New Mexico, Albuquerque, New Mexico.

It can easily be verified that the sequence consists of those years when the United States has celebrated Independence Day (July 4) on a Sunday.

II. Solution by Jeffrey Shallit, Wynnewood, Pennsylvania.

According to the World Almanac, the sequence is characterized by the years in which Christmas falls on a Saturday.

Also solved by the Proposer.

CLOSE FACTORING

B-341 Proposed by Peter Lindstrom, Genesee Community College, Batavia, New York.

Prove that the product $F_{2n}F_{2n+2}F_{2n+4}$ of three consecutive Fibonacci numbers with even subscripts is the product of three consecutive integers.

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

It is well known (see, for example I_{19} in Hoggatt's *Fibonacci and Lucas Numbers*) that

$$F_{n-k}F_{n+k} - F_n^2 = (-1)^{n+k+1}F_k^2.$$

Therefore, replacing n by $2n+2$ and letting $k = 2$, one obtains

$$F_{2n}F_{2n+4} = F_{2n+2}^2 - 1 = (F_{2n+2} - 1)(F_{2n+2} + 1).$$

Thus

$$F_{2n}F_{2n+2}F_{2n+4} = (F_{2n+2} - 1)(F_{2n+2})(F_{2n+2} + 1).$$

Also solved by Gerald Bergum, Richard Blazej, Wray Brady, Michael Brozinsky, Paul S. Bruckman, Herta T. Freitag, Dinh Thế Hung, H. Turner Laquer, Graham Lord, Carl F. Moore, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

PERFECT CUBES

B-342 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Prove that

$$2L_{n-1}^3 + L_n^3 + 6L_{n+1}^2L_{n-1}$$

is a perfect cube for $n = 1, 2, \cdots$.

Solution by Graham Lord, Université Laval, Québec, Canada.

$$2L_{n-1}^3 + L_n^3 + 6L_{n+1}^2L_{n-1} = 2L_{n-1}^3 + (L_{n+1} - L_{n-1})^3 + 6L_{n+1}^2L_{n-1} = (L_{n+1} + L_{n-1})^3 = (5F_n)^3.$$

Also solved by Gerald Bergum, George Berzsenyi, Wray Brady, Paul S. Bruckman, Herta T. Freitag, Dinh Thế Hung, H. Turner Laquer, John W. Milsom, Carl F. Moore, C. B. A. Peck, James F. Pope, Bob Prielipp, Jeffrey Shallit, Sahib Singh, David Zeitlin, and the Proposer.

CLOSED FORM

B-343 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a simple expression for

$$\sum_{k=1}^{n} [F_{2k-1}F_{2(n-k)+1} - F_{2k}F_{2(n-k+1)}] \, .$$

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

$$F_{2k-1}F_{2n+1-2k} - F_{2k}F_{2n+2-2k} = \frac{1}{5}[L_{2n} + L_{4k-2n-2} - L_{2n+2} + L_{4k-2n-2}] = \frac{1}{5}[2L_{4k-2n-2} - L_{2n+1}],$$

$$\sum_{k=1}^{n} [F_{2k-1}F_{2(n-k)+1} - F_{2k}F_{2(n-k+1)})] = \frac{2}{5}\sum_{1}^{n} L_{4k-2n-2} - \frac{n}{5}L_{2n+1}$$

$$= \frac{2}{5}[F_{2(2k-1)-2r-2}]_1^{n+1} - \frac{n}{5}L_{2n+1} = \frac{2}{5}[F_{2n} - F_{-2n}] - \frac{n}{5}L_{2n+1} = \frac{1}{5}[4F_{2n} - nL_{2n+1}] \, .$$

Also solved by Gerald Bergum, Paul S. Bruckman, Herta T. Freitag, H. Turner Laquer, C. B. A. Peck, Bob Prielipp, Sahib Singh, and the Proposer.

AVERAGING GIVES G. P.'S

B-344 Proposed by Frank Higgins, Naperville, Illinois.

Let c and d be real numbers. Find $\lim_{n\to\infty} x_n$, where x_n is defined by

$$x_1 = c, \quad x_2 = d, \quad \text{and} \quad x_{n+2} = (x_{n+1} + x_n)/2 \quad \text{for} \quad n = 1, 2, 3, \cdots.$$

Solution by Sahib Singh, Clarion State College, Clarion, Pennsylvania.

It is easy to see that $x_{2n+1} - x_1$ and $x_{2n} - x_2$ are both geometric progressions with ¼ as common ratio. Thus $\lim_{n\to\infty} x_n = (c + 2d)/3$.

Also solved by Gerald Bergum, George Berzsenyi, Wray Brady, Michael Brozinsky, Paul S. Bruckman, Charles Chouteau, Herta T. Freitag, Ralph Garfield, Dinh Thế Hung, H. Turner Laquer, jointly by Robert McGee and Gerald Satlow and Patricia Cianfero, Carl F. Moore, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Gregory Wulczyn, David Zeitlin, and the Proposer.

ANOTHER LIMIT

B-345 Proposed by Frank Higgins, Naperville, Illinois.

Let $r > s > 0$. Find $\lim_{n\to\infty} P_n$, where P_n is defined by

$$P_1 = r + s \quad \text{and} \quad P_{n+1} = r + s - (rs/P_n) \quad \text{for} \quad n = 1, 2, 3, \cdots.$$

Solution by Wray Brady, Knoxville, Tennessee.

One can establish by an induction that

$$P_n = (r^{n+1} - s^{n+1})/(r^n - s^n)$$

from which it follows that $P_n \to r$ as $n \to \infty$.

Also solved by Gerald Bergum, George Berzsenyi, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, H. Turner Laquer, jointly by Robert McGee and Gerald Satlow, Jeffrey Shallit, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin and the Proposer.

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