SEQUENCES ASSOCIATED WITH t-ARY CODING OF FIBONACCI'S RABBITS

H. W. GOULD and J. B. KIM
West Virginia University, Morgantown, West Virginia 26506
and
V. E. HOGGATT, JR.
San Jose State University, San Jose, California 95192

The object of this note is to point out a curious kind of sequence which arises in connection with a binary coding of the tree diagram for the production of rabbits by Fibonacci's recurrence.

At the left below is a standardized way of drawing the usual Fibonacci rabbit tree. At the right is a binary code for each level. The code is assigned by a very simple rule. On each level, a single segment  is coded by 0 and a branched segment  is coded by 1. It is clear that this establishes a unique binary coding for each level of the Fibonacci rabbit tree (or any other tree for that matter). We suspect that this is not a new idea, but do not have a reference.

TREE CODING

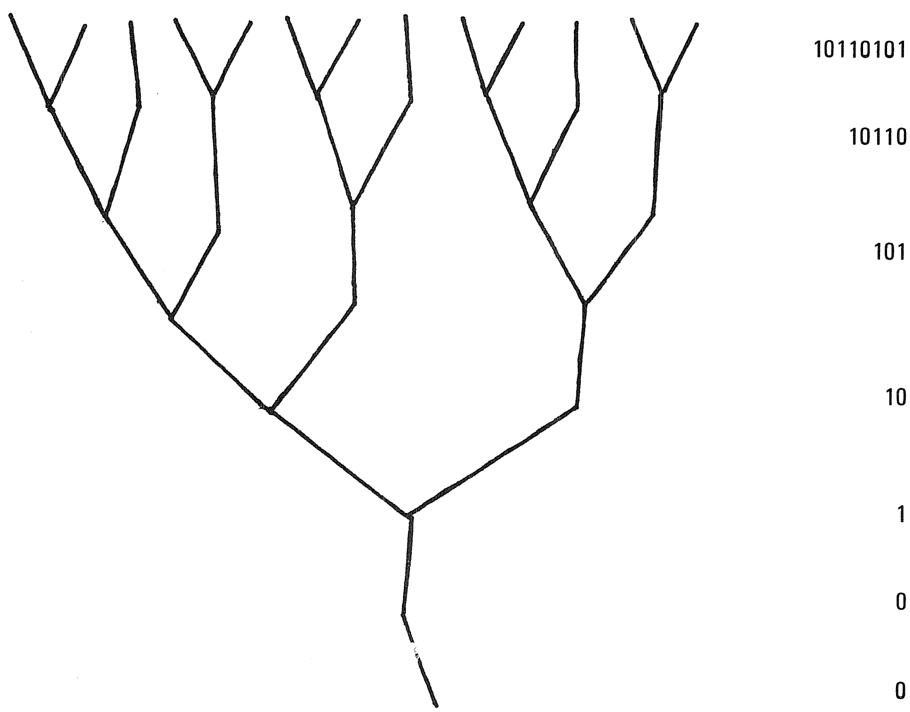


Fig. 1 Coding Numbers

In the next table we give a summary of initial values of the binary coding, first in base 2 and then converted into base ten. In each case notice that the coding number for a given level can be expressed in terms of the coding for the two previous levels.

Table 1
Coding Numbers and Recurrences

n	$(C_n)_2$	$(C_n)_{10} = A_n$
1	0	0
2	0	0
3	1	1
4	10	$2 = 2^1(1) + 0$
5	101	$5 = 2^1(2) + 1$
6	10110	$22 = 2^2(5) + 2$
7	10110101	$181 = 2^3(22) + 5$
8	1011010110110	$5{,}814 = 2^5(181) + 22$
9	101101011011010110101	$1{,}488{,}565 = 2^8(5{,}814) + 181$
10		$12{,}194{,}330{,}294 = 2^{13}(1{,}488{,}565) + 5{,}814$

We put $(C_n)_2$ for the coding number in binary form, and $(C_n)_{10}$ or A_n for the coding number expressed in base ten.

It is evident from the formation of the rabbit tree that the base ten coding numbers satisfy the recurrence

$$A_{n+2} = 2^{F_{n-1}}A_{n+1} + A_n, \qquad n \geq 2, \tag{1}$$

where F_n is the ordinary Fibonacci sequence, $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 0$, $F_1 = 1$. Again, from the law of formation it is evident that $(C_n)_2$ has exactly F_{n-1} digits. Thus also

$$2^{F_{n-1}} > A_n \geq 2^{F_{n-1}-1}, \qquad \text{for} \quad n \geq 3. \tag{2}$$

Formula (1), together with initial values defines the sequence A_n uniquely. Starting with the sequence A_n we may recover the Fibonacci numbers from the formula

$$F_n = \log_2 \frac{A_{n+3} - A_{n+1}}{A_{n+2}}. \tag{3}$$

Special sums involving the sequence A_n may be found in closed form. From (1) we can get almost at once

$$A_{n+3} + A_{n+2} - 1 = \sum_{k=1}^{n} 2^{F_k}A_{k+2}, \qquad n \geq 1. \tag{4}$$

Multiply each side of (1) by 2^{F_n} and use the fact that $F_n + F_{n-1} = F_{n+1}$. We find then

$$2^{F_n}A_{n+2} = 2^{F_{n+1}}A_{n+1} + 2^{F_n}A_n, \qquad n \geq 2, \tag{5}$$

and this form of the recurrence is the clue to the proof of the next formula:

$$\sum_{k=2}^{n} (-1)^k 2^{F_k}A_{k+2} = (-1)^n 2^{F_{n+1}}A_{n+1}, \qquad n \geq 2. \tag{6}$$

We have not found a generating function for A_n and this is posed as a research problem for the reader.

We have also not found the sequence A_n in Sloane's book [2]. Does any reader know any previous appearance of A_n?

The process by which we have obtained A_n is not restricted to the standard Fibonacci sequence. Here is another example yielding a different sequence with the same behavior. Define a third-order recurrent sequence by the recurrence

(7) $$G_{n+1} = G_n + G_{n-2}, \quad \text{with} \quad G_1 = G_2 = G_3 = 1.$$

The reader may draw the corresponding rabbit tree and verify that the coding numbers and recurrence values in the next table are correct.

Table 2
Coding Numbers for G_n

n	G_n	$(D_n)_2$	$(D_n)_{10} = B_n$
1	1	0	0
2	1	0	0
3	1	0	0
4	2	1	1
5	3	10	2 = 2(1) + 0
6	4	100	4 = 2(2) + 0
7	6	1001	9 = 2(4) + 1
8	9	100110	38 = 4(9) + 2
9	13	100110100	308 = 8(38) + 4
10	19	1001101001001	4,937 = 16(308) + 9
11	28		158,022 = 64(4,937) + 38

Here it is evident that the law of formation is

(8) $$B_{n+3} = 2^{G_{n-1}}B_{n+2} + B_n, \qquad n \geqslant 3.$$

Again sums such as (4) and (6) can be established.

It appears that the behavior of these sequences can be predicted to follow in similar fashion for other recurrent sequences for which we can draw a suitable tree.

Recalling that the Lucas numbers are related to the Fibonacci numbers by the formula $L_n = F_{n-1} + F_{n+1}$, we see that we can devise a Lucas rabbit tree by adding together two Fibonacci trees. We can call this method allowing twins to occur once in the Fibonacci tree. It is then evident that the binary coding must correspond to

(9) $$(E_n)_2 = 2^{F_{n-3}}(C_n)_2 + (C_{n-2})_2 \;,$$

and we have the associated sequence $H_n = (E_n)_{10}$ satisfying

(10) $$H_n = 2^{F_{n-3}}A_n + A_{n-2}$$

in terms of our original coding. The corresponding Lucas rabbit tree is exhibited on the following page.

Because we start the twinning at level 3, we have defined $(E_3)_2 = 1$ and $H_3 = (E_3)_{10} = 1$ which is consistent with $H_3 = 2^{F_0}A_3 + A_1 = 1 + 0 = 1.$

We make some further remarks about the coding of the original Fibonacci rabbit tree. The sequence defined by

(11) $$U_n = 2^{F_{n-1}} - 1$$

satisfies

(12) $$U_{n+2} = 2^{F_{n-1}}U_{n+1} + U_n,$$

because

$$U_{n+2} = 2^{F_{n+1}} - 1 = 2^{F_{n-1}}(2^{F_n} - 1) + 2^{F_{n-1}} - 1 = 2^{F_{n-1}}U_{n+1} + U_n,$$

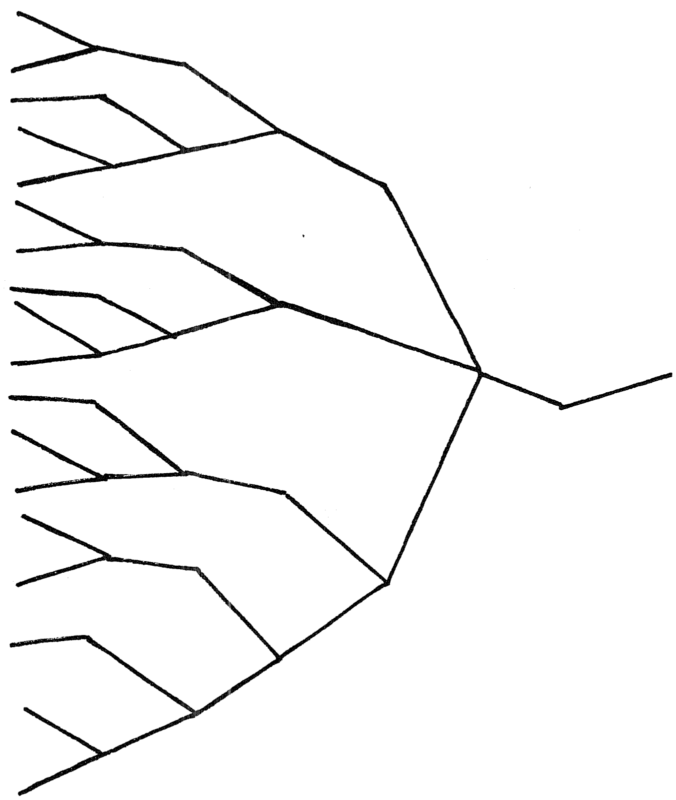
and so U_n is another solution of the equation (1).

In fact U_n and A_n can be found as numerator and denominator, respectively, of the partial convergents of the continued fraction

(13) $$1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{8+}\,\frac{1}{32+}\,\frac{1}{256+}\cdots,$$

where the terms are defined from $2^{F_{n-1}}$. Thus the partial convergents of (13) turn out to be:

TREE



CODING

$(E_n)_2$	$H_n = (E_n)_{10}$
10110101101	1453
1011010	90
1011	11
100	4
1	1
0	0
0	0

Fig. 2 Coding of Lucas Rabbit Tree

(14) $$\frac{U_n}{A_n} = 1, \frac{3}{2}, \frac{7}{5}, \frac{31}{22}, \frac{255}{181}, \frac{8191}{5814}, \cdots$$

The sequence of values

$$\frac{3}{2} = 1.5$$

$$\frac{7}{5} = 1.4$$

$$\frac{31}{22} = 1.4090909\cdots$$

$$\frac{255}{181} = 1.408839779\cdots$$

$$\frac{8191}{5814} = 1.408840729\cdots$$

$$\frac{2097151}{1488565} = 1.408840729\cdots$$

suggests that there exists a limit of the form

(15) $$\lim_{n\to\infty} \frac{U_n}{A_n} = \lim_{n\to\infty} \frac{2^{F_{n-1}}}{A_n} = 1.40884073\cdots$$

which would be somewhat analogous to the well-known limit

(16) $$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} = \frac{2.236068+1}{2} = 1.6180339^{+}\ .$$

Formula (15) would also yield the asymptotic formula

(17) $$A_n \sim (0.709803442\cdots)2^{F_{n-1}} \quad \text{as } n\to\infty\ .$$

Davison [1] has just proved that with $a = (1+\sqrt{5})/2$ then

(18) $$T(a) = \sum_{n=1}^{\infty} \frac{1}{2^{[na]}} = \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{4+}\ \frac{1}{8+}\ \frac{1}{32+}\ \cdots$$

is transcendental. This remarkable result combines two things, the equivalence of the series and continued fraction, and the fact that the number so defined *is* transcendental. $T(a)$ is the reciprocal of the continued fraction in (13), so we have the transcendental limit

(19) $$\lim_{n\to\infty} \frac{A_n}{U_n} = \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{4+}\ \frac{1}{8+}\ \frac{1}{32+}\ \cdots = \sum_{n=1}^{\infty} \frac{1}{2^{[na]}} = 0.709803442\cdots$$

with $a = (1+\sqrt{5})/2$, and where square brackets denote the greatest integer function.

So far we have restricted our attention to binary coding. We return now to Table 1 and consider ternary coding. Actually what we do is to interpret the numbers $(C_n)_2 = (C_n)_3$ as if they were in ternary rather than binary form. Translating the ternary code to base ten, and writing $(C_n)_3 = A_n(3)$, we get the following sequence of numbers:

(20) $$A_n(3) = 0, 0, 1, 3, 10, 93, 2521, 612696, 4019900977, \cdots$$

and this sequence enjoys most of the properties belonging to the original sequence $A_n = A_n(2)$ derived from binary coding. Thus $2521 = 3^3(93) + 10$, $612696 = 3^5(2521) + 93$, etc., and in general

(21) $$A_{n+2}(3) = 3^{F_{n-1}}A_{n+1}(3) + A_n(3), \qquad n \geq 2.$$

As a matter of fact it is just as easy to consider the original coding with 0's and 1's as being t-ary coding, i.e., numbers in base t, where $t = 2, 3, 4, \cdots$. We write $(C_n)_t = A_n(t)$ for this form of the sequence. It is not difficult to see then that the formulas we developed for the binary case become in general:

(22) $$A_{n+2}(t) = t^{F_{n-1}}A_{n+1}(t) + A_n(t), \qquad n \geqslant 2,$$

(23) $$t^{F_{n-1}} > A_n(t) \geqslant t^{F_{n-1}-1}, \qquad n \geqslant 3,$$

(24) $$F_n = \log_t \frac{A_{n+3}(t) - A_{n+1}(t)}{A_{n+2}(t)},$$

(25) $$A_{n+3}(t) + A_{n+2}(t) - 1 = \sum_{k=1}^{n} t^{F_k}A_{k+2}(t), \qquad n \geqslant 1,$$

(26) $$t^{F_n}A_{n+2}(t) = t^{F_{n+1}}A_{n+1}(t) + t^{F_n}A_n(t), \qquad n \geqslant 2,$$

(27) $$\sum_{k=2}^{n} (-1)^k t^{F_k}A_{k+2}(t) = (-1)^n t^{F_{n+1}}A_{n+1}(t), \qquad n \geqslant 2,$$

and in place of the sequence U_n we have the corresponding extension

(28) $$U_n(t) = t^{F_{n-1}} - 1,$$

which satisfies the recurrence

(29) $$X_{n+2}(t) = t^{F_{n-1}}X_{n+1}(t) + X_n(t),$$

as an extension of (1).

We also have an asymptotic result of the form

$$A_n(t) \sim K \cdot t^{F_{n-1}}, \qquad n \to \infty.$$

We shall find K in terms of continued fractions.

The continued fraction (13) with partial convergents (14) has a very interesting form in the general t-ary case:

(30) $$\frac{U_n(t)}{A_n(t)} = (t-1) + \frac{t-1}{t^1+}\,\frac{1}{t^1+}\,\frac{1}{t^2+}\,\frac{1}{t^3+}\cdots\frac{1}{t^{F_{n-3}}}.$$

For $t = 3$ we have the case

(31) $$2 + \frac{2}{3+}\,\frac{1}{3+}\,\frac{1}{9+}\,\frac{1}{27+}\,\frac{1}{243+}\cdots = 2.602142009\cdots$$

The reciprocal of this is $0.3842987802\cdots$, and it is now remarkable to note that if we extend the series of Davison (18) in the obvious way, we find that

(32) $$\sum_{n=1}^{\infty} \frac{1}{3^{[n\alpha]}} = \frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^6} + \frac{1}{3^8} + \frac{1}{3^9} + \frac{1}{3^{11}} + \cdots = 0.3842987802\cdots$$

and this is correct to at least as many decimals as shown since we have calculated the sum to 20 terms and the 21^{st} term adds only about 1.798865×10^{-16} to this.

It is natural to conjecture that Davison's theorem can be extended to show that this number also is transcendental and moreover that the limit of (30) as $n \to \infty$ is probably transcendental for every natural number $t \geqslant 2$.

Some of the first few partial convergents of (31) are:

(33) $$2,\ \frac{8}{3},\ \frac{26}{10},\ \frac{242}{93},\ \frac{6560}{2521},\ \frac{1594322}{612696},\ \cdots.$$

The general theorem which we claim is that for the continued fraction in (30),

(34) $$\left(\sum_{n=1}^{\infty} \frac{1}{t^{[na]}}\right)^{-1} = \lim_{n\to\infty} \frac{U_n(t)}{A_n(t)} = (t-1) + \frac{t-1}{t+}\,\frac{1}{t+}\,\frac{1}{t^2+}\,\frac{1}{t^3+}\,,$$

where the exponents in the continued fraction are the successive Fibonacci numbers.

The first few partial convergents of the general continued fraction in (30) are:

$$\frac{U_4(t)}{A_4(t)} = \frac{t^2-1}{t}, \quad \frac{U_5(t)}{A_5(t)} = \frac{t^3-1}{t^2+1}, \quad \frac{U_6(t)}{A_6(t)} = \frac{t^5-1}{t^4+t^2+t},$$

$$\frac{U_7(t)}{A_7(t)} = \frac{t^8-1}{t^7+t^5+t^4+t^2+1}, \quad \frac{U_8(t)}{A_8(t)} = \frac{t^{13}-1}{t^{12}+t^{10}+t^9+t^7+t^5+t^4+t^2+t},$$

etc., where, of course, the numerator is $t^{F_{n-1}} - 1$, and the exponents of the t's in the denominator are precisely the powers of 2 appearing in the original binary coding of the rabbit tree as given in Fig. 1 or Table 1.

The first 50 values of $[na]$ for use in writing out the series (34) are: 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, 30, 32, 33, 35, 37, 38, 40, 42, 43, 45, 46, 48, 50, 51, 53, 55, 56, 58, 59, 61, 63, 64, 66, 67, 69, 71, 72, 74, 76, 77, 79, 80. This agrees with sequence No. 917 in Sloane [2], where it is called a Beatty sequence because of the fact that $a_n = [na]$ and $b_n = [nb]$, where a and b are irrational with $1/a + 1/b = 1$ makes a_n and b_n disjoint subsequences of the natural numbers whose union is precisely the set of all natural numbers. Such sets are called complementary sequences.

Relations (30) and (34) may be put in more attractive form. Dividing each side of (30) by $t-1$ we get

(35) $$\frac{U_n(t)}{(t-1)A_n(t)} = 1 + \frac{1}{t+}\,\frac{1}{t+}\,\frac{1}{t^2+}\,\frac{1}{t^3+}\,\frac{1}{t^5+}\cdots\frac{1}{t^{F_{n-3}}},$$

and taking reciprocals on both sides we find

(36) $$\frac{(t-1)A_n(t)}{U_n(t)} = \frac{1}{1+}\,\frac{1}{t+}\,\frac{1}{t+}\,\frac{1}{t^2+}\cdots\frac{1}{t^{F_{n-3}}}.$$

Then the limiting case (34) becomes more elegantly

(37) $$(t-1)\sum_{n=1}^{\infty}\frac{1}{t^{[na]}} = \lim_{n\to\infty}\frac{(t-1)A_n(t)}{U_n(t)} = \frac{1}{1+}\,\frac{1}{t+}\,\frac{1}{t+}\,\frac{1}{t^2+}\,\frac{1}{t^3+}\,\frac{1}{t^5+}\,\frac{1}{t^8+\cdots},$$

apparently valid for all real $t > 1$.

Although the series diverges when $t = 1$, still the continued fraction makes sense, giving the familiar special case

(38) $$\lim_{t\to 1}\lim_{n\to\infty}\frac{(t-1)A_n(t)}{U_n(t)} = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots = \frac{1+\sqrt{5}}{2}.$$

For $t = 1$, the sequence $A_n(1) = F_n$ so that we have in the general sequence an extension of the Fibonacci sequence.

Let us now make the definition

(39) $$T(x,t) = \sum_{n=1}^{\infty}\frac{1}{t^{[nx]}}$$

for arbitrary real $t > 1$ and real $x > 0$.

This function has interesting properties, some of which we shall exhibit here. Take $x = a - 1 = 1/a$, a being as defined before. Then the sequence of values of $[na - n] = [na] - n$ begins: 0, 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 9, 10, 11, 11, 12, 12, 13, 14, 14, 15, 16, 16, 17, 17, 18, 19, 19, 20, ⋯. It does not seem to be tabulated in Sloane [2]. Taking $t = 2$, one finds that $T(a-1, 2) = 2.7098016\cdots$ and it seems evident that in fact $T(a-1, 2) = 2 + T(a,2)$. For $t = 3$ we find that

$$T(a-1, 3) = 1.884298779\cdots = 1.5 + 0.384298779\cdots = 3/2 + T(a,3).$$

For $t = 7$, we find that

$$T(a-1, 7) = 1.312864454\cdots = 7/6 + T(a, 7).$$

The general result appears to be

$$T(a-1, t) = \frac{t}{t-1} + T(a, t), \qquad t > 1. \tag{40}$$

This appears to depend on the value of a being $(1+\sqrt{5})/2$. Indeed,

$$T(\pi, 7) = 2.923976609\cdots \quad \text{and} \quad T(\pi - 1, 7) = 0.02083333\cdots$$

while $7/6 = 1.16666\cdots$ so that (40) does not hold.

Here is another numerical result that may be of some interest:

$$T(a,a) = \sum_{n=1}^{\infty} \frac{1}{a^{[na]}} = 1.100412718\cdots. \tag{41}$$

Some of the partial convergents from the continued fraction are:

$$\frac{A_6(a)}{U_6(a)} = \frac{11.09016995}{10.09016995} = 1.099106358\cdots.$$

Note that $(11/10) = 1.1$;

$$\frac{A_7(a)}{U_7(a)} = \frac{50.59674778}{45.97871383} = 1.10043852\cdots.$$

Note that $(50.6/46) = 1.1$;

$$\frac{A_8(a)}{U_8(a)} = \frac{572.2107019}{520.0019205} = 1.10041267\cdots.$$

Note that $(572/520) = 1.1$.

It is interesting to note that $T(a,a)$ is just slightly larger than 1.1, suggesting this as a dominant term.

Here is still another numerical example of (40): Let $e = 2.7182818\cdots$.

$$T(a, e) = 0.438943611\cdots, \quad T(a-1, e) = 2.020920317\cdots, \quad e/(e-1) = 1.581976707\cdots,$$

so that

$$T(a, e) + e/(e-1) = 2.020920318 = T(a-1, e)$$

as closely as we could compute the numbers.

REFERENCES

1. John L. Davison, "A Series and Its Associated Continued Fraction," *Proc. Amer. Math. Soc.*, 63 (1977), pp. 29–32.

2. N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, New York — London, 1973. Also supplements sent by the author.

****