GENERALIZED QUATERNIONS OF HIGHER ORDER

I. L. IAKIN
University of New England, Armidale, Australia

In a previous article [2], we conjectured that the idea of a quaternion with quaternion components could be extended to include higher order quaternions. The purpose of this article is to investigate this concept and to obtain further generalizations of the results in [2].

PROPERTIES

Firstly, to be able to denote higher order quaternions, we need to introduce an operator notation. Thus for λ a positive integer we define the *quaternions of order* λ, after λ operations, as:

(1) $$\Omega^\lambda W_n = \Omega(\Omega(\Omega\cdots(\Omega W_n)\cdots)) = \Omega^{\lambda-1}W_n + i\Omega^{\lambda-1}W_{n+1} + j\Omega^{\lambda-1}W_{n+2} + k\Omega^{\lambda-1}W_{n+3}$$

(2) $$\Delta^\lambda W_n = \Delta(\Delta(\Delta\cdots(\Delta W_n)\cdots)) = \Delta^{\lambda-1}W_n + iq\Delta^{\lambda-1}W_{n-1} + jq^2\Delta^{\lambda-1}W_{n-2} + kq^3\Delta^{\lambda-1}W_{n-3}$$

where we also define

(3) $$\Omega^0 W_n = W_n,\quad \Delta^0 W_n = W_n,\quad \Omega^1 W_n = \Omega W_n,\quad \Delta^1 W_n = \Delta W_n$$

and the quaternion vectors i,j,k have the following properties

(4) $$i^2 = j^2 = k^2 = -1,\quad ij = -ji = k,\quad jk = -kj = i,\quad ki = -ik = j$$

and where from Horadam [1] we have that for integers $a,b,p,q,$

(5a) $$W_n \equiv W_n(a,b;p,q)$$

$$W_n = pW_{n-1} - qW_{n-2}\quad \text{for } n \geqslant 2$$

$$W_0 = a,\quad W_1 = b$$

(5b) $$U_n \equiv W_n(1,p;p,q)$$

(5c) $$V_n \equiv W_n(2,p;p,q)$$

(5d) $$\epsilon = pab - qa^2 - b^2 .$$

Thus we see from (1), (3), (5b) and (5c) that for $\lambda = 1$ we obtain the special cases 1(a), 1(b) and 1(c) of [2], while $\lambda = 2$ gives us 7(a) and 7(b) of [2]. Equation (11) of [2] is obtained from (2) and (3) for $\lambda = 1$.

We can now combine the operators Ω and Δ to define quaternions of the type $\Omega\Delta W_n$ and $\Delta\Omega W_n$, i.e.,

(6) $$\Omega\Delta W_n = \Omega(\Delta W_n) = \Delta W_n + i\Delta W_{n+1} + j\Delta W_{n+2} + k\Delta W_{n+3}$$

(7) $$\Delta\Omega W_n = \Delta(\Omega W_n) = \Omega W_n + iq\Omega W_{n-1} + jq^2\Omega W_{n-2} + kq^3\Omega W_{n-3} .$$

If we expand (6) and (7) we see that

$$\Omega\Delta W_n \neq \Delta\Omega W_n .$$

Since quaternion vector multiplication is non-commutative we also know that

$$i\cdot\Omega W_m\Omega W_n \neq \Omega W_m\cdot i\cdot\Omega W_n \neq \Omega W_m \Omega W_n\cdot i .$$

To overcome some of the problems associated with calculations involving higher order quaternions, resulting from the failure of the commutative law for quaternion multiplication, we introduce two new operators, namely Ω^* and Δ^*. We thus define

(8) $$\Omega^*\Omega W_n = \Omega^*(\Omega W_n) = \Omega W_n + \Omega W_{n+1}\cdot i + \Omega W_{n+2}\cdot j + \Omega W_{n+3}\cdot k$$

(9) $$\Delta^*\Delta W_n = \Delta^*(\Delta W_n) = \Delta W_n + q\Delta W_{n-1}\cdot i + q^2\Delta W_{n-2}\cdot j + q^3\Delta W_{n-3}\cdot k$$

Hence we see that the operators Ω^* and Δ^* are the same as the operators Ω and Δ except that they create quaternions by post-multiplication of the quaternion vectors. Obviously

$$\Omega^* W_n = \Omega W_n$$

and since, say

$$\Delta\Omega^* W_n = \Delta(\Omega^* W_n) = \Delta\Omega W_n$$

it follows that the star operators are only meaningful when applied to the L.H.S. of quaternions of order $\geqslant 1$.

If we now expand the R.H.S. of Eq. (8) we see that we have result 8(a) of [2], i.e.,

(10) $$\Omega^*\Omega W_n = \Omega^2 W_n .$$

Similarly from (9) it follows that

(11) $$\Delta^*\Delta_n = \Delta^2 W_n .$$

We leave it to the reader to show, by expanding, that

(12) $$\Omega\Delta W_n = \Delta^*\Omega W_n$$

(13) $$\Delta\Omega W_n = \Omega^*\Delta W_n$$

and to prove the associative laws for the operators, e.g.,

(14) $$(\Omega\Delta)\Omega W_n = \Omega(\Delta\Omega)W_n, \qquad (\Delta\Omega)\Delta W_n = \Delta(\Omega\Delta)W_n .$$

Now for μ a positive integer we know

$$\begin{aligned}
\Omega^*\Omega^\mu W_n &= \Omega^*(\Omega\Omega^{\mu-1})W_n && \text{(by (1))}\\
&= (\Omega^*\Omega)\Omega^{\mu-1}W_n && \text{(Associative laws)}\\
&= \Omega^2\Omega^{\mu-1}W_n && \text{(by (10))}
\end{aligned}$$

(15) $$\Omega^*\Omega^\mu W_n = \Omega^{\mu+1}W_n \qquad \text{(by (1))}.$$

If we replace Ω by Δ in the above proof we obtain the result

(16) $$\Delta^*\Delta^\mu W_n = \Delta^{\mu+1}W_n .$$

Next, induction on μ produces the results

(17) $$\Omega\Delta^\mu W_n = (\Delta^*)^\mu\Omega W_n$$

(18) $$\Delta\Omega^\mu W_n = (\Omega^*)^\mu\Delta W_n .$$

Using the above results and induction on λ we can prove the following

(19) $$\Lambda^\lambda\Omega W_n = \Omega^*\Delta^\lambda W_n$$

(20) $$\Omega^\lambda\Delta W_n = \Delta^*\Omega^\lambda W_n$$

(21) $$(\Omega^*)^\lambda\Omega^\mu W_n = \Omega^{\lambda+\mu}W_n$$

(22) $$(\Delta^*)^\lambda\Delta^\mu W_n = \Delta^{\lambda+\mu}W_n$$

(23) $$\Omega^\lambda\Delta^\mu W_n = (\Delta^*)^\mu\Omega^\lambda W_n$$

(24) $$\Delta^\lambda\Omega^\mu W_n = (\Omega^*)^\mu\Delta^\lambda W_n$$

EXTENDED GENERALIZED RESULTS

In this section we extend some of the identities given in Iakin [2]. We commence by proving the generalization of Eq. (10) of [2].

(25) $$\Omega^\lambda U_{-n} = -q^{-n+1}\Delta^\lambda U_{n-2} .$$

Proof. We prove this result using induction on λ. For $\lambda = 1$ we have Eq. (10) of [2]. Assume the result is true for $\lambda = h$, i.e.,

$$\Omega^h U_{-n} = -q^{-n+1}\Delta^h U_{n-2} .$$

Now for $\lambda = h+1$ we have from (1)

$$\Omega^{h+1}U_{-n} = \Omega^h U_{-n} + i\Omega^h U_{-n+1} + j\Omega^h U_{-n+2} + k\Omega^h U_{-n+3}$$

which becomes on using the assumption

$$\begin{aligned}\Omega^{h+1}U_{-n} &= -q^{-n+1}\Delta^h U_{n-2} - iq^{-n+2}\Delta^h U_{n-3} - jq^{-n+3}\Delta^h U_{n-4} - kq^{-n+4}\Delta^h U_{n-5}\\ &= -q^{-n+1}(\Delta^h U_{n-2} + iq\Delta^h U_{n-3} + jq^2\Delta^h U_{n-4} + kq^3\Delta^h U_{n-5}) = -q^{-n+1}\Delta^{h+1}U_{n-2} \text{ (by (2))}\end{aligned}$$

Since the result holds for $\lambda = 1$ and is true for $\lambda = h+1$ providing it is true for $\lambda = h$, then by the principle of induction the result holds for all positive integer values of λ.

Similarly we can show by induction on λ that

(26) $$\Omega^\lambda V_{-n} = q^{-n}\Delta^\lambda V_n$$

(27) $$\Omega^\lambda W_{-n} = q^{-n}(a\Delta^\lambda U_n - b\Delta^\lambda U_{n-1})$$

(28) $$\Omega^\lambda W_{n+r} + q^r\Omega^\lambda W_{n-r} = V_r\Omega^\lambda W_n .$$

After a lengthy proof using induction on $\lambda + \mu$ we have

(29) $$\Omega^{\lambda+\mu}W_{m+n} = \Omega^\lambda W_m\Omega^\mu U_n - q\Omega^\lambda W_{m-1}\Omega^\mu U_{n-1}$$

for which we obtain the special cases

(30) $$2\Omega^{\lambda+\mu}U_{m+n-1} = \Omega^\lambda U_{m-1}\Omega^\mu V_n + \Omega^\lambda V_m\Omega^\mu U_{n-1}$$

(31) $$2\Omega^{\lambda+\mu}V_{m+n} = \Omega^\lambda V_m\Omega^\mu V_n + d^2\Omega^\lambda U_{m-1}\Omega^\mu U_{n-1}$$

where $d^2 = p^2 - 4q$.

If we again use induction on $\lambda + \mu$ we can arrive at

(32) $$\Omega^\lambda W_m\Omega^\mu W_n - q\Omega^\lambda W_{m-1}\Omega^\mu W_{n-1} = a\Omega^{\lambda+\mu}W_{m+n} + (b-pa)\Omega^{\lambda+\mu}W_{m+n-1}$$

(33) $$\Omega^\lambda W_{m+1}\Omega^\mu W_{n+1} - q^2\Omega^\lambda W_{m-1}\Omega^\mu W_{n-1} = b\Omega^{\lambda+\mu}W_{m+n+1} + (b-pa)q\Omega^{\lambda+\mu}W_{m+n-1} .$$

Now letting $m = n$ and $\lambda = \mu$ in both (32) and (33) gives us

(34) $$(\Omega^\lambda W_n)^2 - q(\Omega^\lambda W_{n-1})^2 = a\Omega^{2\lambda}W_{2n} + (b-pa)\Omega^{2\lambda}W_{2n-1}$$

(35) $$(\Omega^\lambda W_{n+1})^2 - (q\Omega^\lambda W_{n-1})^2 = b\Omega^{2\lambda}W_{2n+1} + (b-pa)q\Omega^{2\lambda}W_{2n-1} .$$

Note that Eqs. (28), (29), (30), (31), (32), (33), (34) and (35) give, as special cases, Eqs. 24(a) and (b), 22(a) and (b), 21(a) and (b), (23), (16) and (17), (20), (18) and (19), respectively.

We now list a set of identities whose proofs we omit due to their length and repetitiveness. We leave it to the reader to prove by induction the following results:

$$W_{n-r}\Omega^{\lambda+\mu}W_{n+r+t}$$

(36) $$= \Omega^\lambda W_n\Omega^\mu W_{n+t} + \epsilon q^{n-r}\Omega^\lambda U_{r-1}\Omega^\mu U_{r+t-1}$$

(37) $$= \Omega^\lambda W_{n+t}\Omega^\mu W_n + \epsilon q^{n-r}\Omega^\lambda U_{r+t-1}\Omega^\mu U_{r-1}$$

$$\Omega^\lambda W_{n-r}\Omega^\mu W_{n+r+t}$$

(38) $$= W_n\Omega^{\lambda+\mu}W_{n+t} + \epsilon q^{n-r}\Delta^\lambda U_{r-1}\Omega^\mu U_{r+t-1}$$

(39) $$= \Omega^\lambda W_n\Omega^\mu W_{n+t} + \epsilon q^{n-r}U_{r-1}\Delta^\lambda\Omega^\mu U_{r+t-1}$$

(40) $$= \Omega^\lambda W_{n+t}\Omega^\mu W_n + \epsilon q^{n-r}U_{r+t-1}\Delta^\lambda\Omega^\mu U_{r-1}$$

(41) $$= W_{n+t}\Omega^{\lambda+\mu}W_n + \epsilon q^{n-r}\Delta^\lambda U_{r+t-1}\Omega^\mu U_{r-1}$$

and finally $\Omega^\lambda W_{m-r+t}\Omega^\mu W_{n+r+s}$

(42) $$= W_{n-r+t}\Omega^{\lambda+\mu}W_{m+r+s} + \epsilon q^{n-r}\Delta^\lambda U_{n-m-1}\Omega^\mu U_{2r-t+s-1}$$

(43) $$= \Omega^\lambda W_{n-r+t}\Omega^\mu W_{m+r+s} + \epsilon q^{n-r}U_{n-m-1}\Delta^\lambda\Omega^\mu U_{2r-t+s-1}$$

(44) $$= \Omega^{\lambda} W_{m+r+s} \Omega^{\mu} W_{n-r+t} + \epsilon q^{n-r} U_{2r-t+s-1} \Delta^{\lambda} \Omega^{\mu} U_{n-m-1}$$

(45) $$= W_{m+r+s} \Omega^{\lambda+\mu} W_{n-r+t} + \epsilon q^{n-r} \Delta^{\lambda} U_{2r-t+s-1} \Omega^{\mu} U_{n-m-1}\ .$$

Putting $\lambda = 1$ and $\mu = 1$ in (36), (39) and (40) gives us, respectively, (13), (26) and (27) of [2], while letting $\lambda = 1$, $\mu = 2$ in (39) and (40) gives, respectively, 28(a) and (b). If, however, we let $t = 0$, $s = 0$, $\lambda = 1$ and $\mu = 1$ in (43) we have as a special case result (29) of [2].

REFERENCES

1. A. F. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers," *The Fibonacci Quarterly,* Vol. 3 (1965), No. 3, pp. 161–175.
2. A.L. Iakin, "Generalized Quaternions with Quaternion Components," *The Fibonacci Quarterly,* 1974 preprint.

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LETTER TO THE EDITOR

16 September 1977

Dear Professor Hoggatt:

In a recent article with Claudia Smith (*The Fibonacci Quarterly,* Vol. 14, No. 4, p. 343), you referred to the question whether a prime p and its square p^2 can have the same rank of apparition in the Fibonacci sequence, and mentioned that Wall (1960) had tested primes up to 10,000 and not found any with this property.

I have recently extended this search and found that no prime up to 1,000,000 (one million) has this property.

My computations in fact test the Lucas sequence for the property

(1) $$L_p \equiv 1 \pmod{p^2} \qquad p = \text{prime}\,.$$

For $p > 5$ this is easily shown to be a necessary and sufficient condition for p and p^2 to have the same rank of apparition in the Fibonacci sequence, because of the identity

(2) $$(L_p - 1)(L_p + 1) = 5F_{p-1}F_{p+1}\,.$$

So far I have shown that the congruence (1) does not hold for any prime less than one million; I hope to extend the search further at a later date.

You may wish to publish these results in *The Fibonacci Quarterly.*

Yours sincerely,

s/ Dr. L. A. G. Dresel
The University of Reading,
Berks, UK