ON A CONJECTURE CONCERNING A SET OF SEQUENCES SATISFYING THE FIBONACCI DIFFERENCE EQUATION

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Let $\alpha = (1+\sqrt{5})/2$ and consider the set of sequences

$$\begin{aligned} S = \{&(1, 1, 2, 3, 5, 8, 13, \cdots),\\ &(2, 4, 6, 10, 16, 26, 42, \cdots),\\ &(4, 7, 11, 18, 29, 47, 76, \cdots),\\ &(6, 9, 15, 24, 39, 63, 102, \cdots),\\ &(7, 12, 19, 31, 50, 81, 131, \cdots), \cdots\}, \end{aligned}$$

where a sequence $u = (u_0, u_1, u_2, \cdots)$ is in S iff it satisfies the conditions

(1) $u_0, u_1, u_2, \cdots$ are positive integers

(2) u satisfies the Fibonacci difference equation $u_n = u_{n-1} + u_{n-2}$ $(n = 2, 3, 4, \cdots)$

(3) there does not exist an integer x such that $|\alpha x - u_1| < ½$

(4) $|\alpha u_1 - u_2| < ½$.

Note that, for given u_1, there must exist an integer u_2 satisfying (4), because of the irrationality of α.

For $n = 0, 1, 2, \cdots$ let $S_n = \{u_n : u \in S\}$. It has been conjectured by Kenneth B. Stolarsky that for any $u \in S$, the value of $u_2 - u_1$ equals the value of either v_1 or v_2 for some $v \in S$. Since $u_2 = u_0 + u_1$, this is equivalent to saying that $S_0 \subset S_1 \cup S_2$. In this paper we prove the stronger result, that $S_0 = S_1 \cup S_2$.

Lemma 1. If $u \in S$ then for all $n = 1, 2, \cdots$

(5) $$\alpha^{1-n}/2 < |\alpha u_{n-1} - u_n| < \alpha^{2-n}/2$$

and

(6) $$\alpha^{1-n}/2 < |\alpha^2 u_{n-1} - u_{n+1}| < \alpha^{2-n}/2.$$

Proof. We first show that, for any u, there is a constant C such that for all $n = 1, 2, \cdots$

(7) $$C = \alpha^n |\alpha u_{n-1} - u_n| .$$

If C_n denotes the value of C given by (7) we have

$$\begin{aligned} C_{n+1} &= \alpha^{n+1}|\alpha u_n - u_{n+1}| = \alpha^n |\alpha^2 u_n - \alpha u_{n+1}|\\ &= \alpha^n |(\alpha+1)u_n - \alpha u_{n+1}| = \alpha^n |\alpha(u_{n+1} - u_n) - u_n|\\ &= \alpha^n |\alpha u_{n-1} - u_n| = C_n . \end{aligned}$$

From (4) we see that $C = \alpha^2 |\alpha u_1 - u_2| < \alpha^2/2$; also we see that $C = \alpha|\alpha u_0 - u_1| < \alpha/2$ since $|\alpha u_0 - u_1|$ cannot equal ½ because it is irrational, and cannot be less than ½ by (3). Combining these inequalities we obtain (5). To prove (6) we simply note that

$$|\alpha^2 u_{n-1} - u_{n+1}| = |(\alpha+1)u_{n-1} - u_n - u_{n-1}| = |\alpha u_{n-1} - u_n| .$$

Lemma 2.

$$\bigcup_{n=1}^{\infty} S_n$$

is the set of positive integers.

Proof. If there were positive integers not in this union, let y be the lowest of these. Since y is not a member of S_1, there exists an integer x such that $|ax - y| < \frac{1}{2}$. Since x is a positive integer less than y, it must lie in $\cup_{n=1}^{\infty} S_n$ and therefore $x = u_n$ for some $u \in S$ and n a positive integer. Since $|au_n - y| < \frac{1}{2}$, $|au_n - u_{n+1}| < \frac{1}{2}$ it follows that $y = u_{n+1} \in S_{n+1}$.

Lemma 3.
$$S_0 \subset S_1 \cup S_2 .$$

Proof. If this result did not hold, because of Lemma 2, there would exist $u,v \in S$ and $n > 2$ such that $u_n = v_0$. By (2) we then find

$$v_2 - u_{n+2} = (v_1 + v_0) - (u_{n+1} + u_n) = v_1 - u_{n+1}$$

so that

$$|(a-1)(v_1 - u_{n+1})| = |(av_1 - v_2) - (au_{n+1} - u_{n+2})| < \tfrac{1}{2} + \tfrac{1}{2}a^{-n} \leqslant \tfrac{1}{2}(1 + a^{-3}),$$

where we have used Lemma 1 to bound $|av_1 - v_2|$ and $|au_{n+1} - u_n|$ and made use of the fact that $n \geqslant 3$. Since $a^{-3} = 2a - 3$ we find

$$|v_1 - u_{n+1}| < \frac{1}{a-1} \cdot \frac{1}{2}(1 + 2a - 3) = 1$$

so that $v_1 = u_{n+1}$. Using Lemma 1 again we find that

$$\tfrac{1}{2} < |av_0 - v_1| = |au_n - u_{n+1}| < a^{1-n}/2 < \tfrac{1}{2},$$

a contradiction.

Lemma 4.
$$S_1 \subset S_0 .$$

Proof. Let $s = +1$ if $au_1 - u_2 > 0$ and -1 otherwise.

By Lemma 1, we have

$$\frac{a^{-1}}{2} < s(au_1 - u_2) < \tfrac{1}{2}.$$

Let $y = u_2 + s$ so that

$$\frac{1}{2} < -s(au_1 - y) < 1 - \frac{a^{-1}}{2}$$

which implies that

$$|au_1 - y| < 1 - \frac{a^{-1}}{2} = 1 - \frac{a^{-1}}{2} = 1 - \frac{a-1}{2} = \frac{a}{2} - \frac{2a-3}{2} < \frac{a}{2}.$$

If there were an x such that $|ax - y| < \frac{1}{2}$, it would follow that

$$|au_1 - ax| < \frac{a+1}{2} = \frac{a^2}{2}$$

which implies

$$|u_1 - x| < \frac{a}{2} < 1$$

so that $u_1 = x$ and $u_2 = y$ which is impossible since $|u_2 - y| = 1$. Hence, no such x exists and therefore $y = v_1$ for some $v \in S$. Thus $|au_1 - v_1| < (a/2)$. We now find

$$|u_1 - v_0| = |u_1 - v_2 + v_1| \leqslant |u_1 - a^{-1}v_1| + |a^{-1}v_1 - v_2 + v_1| = (a-1)|au_1 - v_1| + |av_1 - v_2|$$
$$< \frac{(a-1)a}{2} + \frac{1}{2} = 1$$

so that $u_1 = v_0 \in S$.

Lemma 5.
$$S_2 \subset S_0 .$$

Proof. Let $s = +1$ if $a^2u_2 - u_4 > 0$ and -1 otherwise. By Lemma 1, we have

$$\frac{a^{-2}}{2} < s(a^2u_2 - u_4) < \frac{a^{-1}}{2}$$

so that if $y = u_4 + s$ then

$$1 - \frac{a^{-1}}{2} < -s(a^2u_2 - y) < 1 - \frac{a^{-2}}{2} .$$

Since

$$1 - \frac{a^{-1}}{2} > 0 \quad \text{and} \quad 1 - \frac{a^{-2}}{2} = \frac{a}{2}$$

it follows that

$$|a^2u_2 - y| < \frac{a}{2}.$$

If there were an integer w such that $|a^2w - y| < ½$ it would follow that

$$a^2|u_2 - w| < \frac{1+a}{2} = \frac{a^2}{2}$$

implying that $w = u_2$ and that $y = u_4$, contradicting the fact that $|y - u_4| = 1$. On the other hand, there is an integer $x = y - u_2$ such that $|ax - y| < ½$ since

$$|ax - y| = |(a-1)y - au_2| = (a-1)|y - a^2u_2| < \frac{a(a-1)}{2} = \frac{1}{2} .$$

The existence of x (and the non-existence of w) satisfying these conditions, implies that $y = v_2$ for some $v \in S$. Thus,

$$|a^2u_2 - v_2| < \frac{a}{2} .$$

We now find

$$|u_2 - v_0| = |u_2 + v_1 - v_2| \leqslant |v_2a^{-2} - u_2| + |v_2(1 - a^{-2}) - v_1|$$
$$= a^{-2}(|v_2 - a^2u_2| + |v_2a - a^2v_1|) < \frac{a^{-1}}{2} + \frac{a^{-1}}{2} = a^{-1} < 1$$

so that $u_2 = v_0 \in S_0$.

Combining the results of Lemmas 3, 4, 5 we have

Theorem. $S_0 = S_1 \cup S_2$.

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A GOLDEN DOUBLE CROSTIC: SOLUTION

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"Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel." J. Kepler. Quotation given in *The Divine Proportion* by Huntley (Dover, New York, 1970, p. 23).

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