RECURRENCES OF THE THIRD ORDER AND RELATED COMBINATORIAL IDENTITIES

L. CARLITZ
Duke University, Durham, North Carolina 27706

1. Let g be a rational integer such that $\Delta = 4g^3 + 27$ is squarefree and let w denote the real root of the equation

(1.1) $$x^3 + gx - 1 = 0 \qquad (g > 1).$$

Clearly w is a unit of the cubic field $Q(w)$.

Following Bernstein [1], put

(1.2) $$w^n = r_n + s_n w + t_n w^2 \qquad (n \geq 0)$$

and

(1.3) $$w^{-n} = x_n + y_n w + z_n w^2 \qquad (n \geq 0).$$

Making use of the theory of units in an algebraic number field, Bernstein obtained some combinatorial identities. He showed that

$$s_n = r_{n+2}, \quad t_n = r_{n+1}, \quad y_n = x_{n-2}, \quad z_n = x_{n-1}$$

and

(1.4) $$\sum_{n=0}^{\infty} r_n u^n = \frac{1+gu^2}{1+gu-u^3}, \qquad \sum_{n=0}^{\infty} x_n u^n = \frac{1}{1-gu^2-u^3}.$$

Moreover, it follows from (1.2) and (1.3) that

(1.5) $$\begin{cases} r_n^2 - r_{n-1}r_{n+1} = x_{n-3} \\ x_n^2 - x_{n-1}x_{n+1} = r_{n+3} \end{cases}.$$

Explicit formulas for r_n and x_n are implied by (1.4). Substituting in (1.5) the combinatorial identities result. Since $\Delta = 4g^3 + 27$ is squarefree for infinitely many values of g, the identities are indeed polynomial identities.

The present writer [2] has proved these and related identities using only some elementary algebra. For example, if we put

$$1 + gx^2 - x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x)$$

and define

$$\sigma_n = \alpha^n + \beta^n + \gamma^n \qquad (\text{all } n)$$

and

$$\rho_n = \begin{cases} r_n & (n \geq 0) \\ x_{-n} & (n \geq 0) \end{cases},$$

then various relations are found connecting these quantities. For example

(1.6) $$\sigma_m \sigma_n = \sigma_{m+n} + \sigma_{m-n}\sigma_{-n} - \sigma_{m-2n}.$$

Each relation of this kind implies a combinatorial identity.

In the present paper we consider a slightly more general situation. Let u, v denote indeterminates and put

$$1 - ux + vx^2 - x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x).$$

We define σ_n by means of

Supported in part by NSF grant GP-37924X.

(1.7) $$\sigma_n = \alpha^n + \beta^n + \gamma^n \qquad \text{(all } n\text{)}$$

and ρ_n by

(1.8) $$\rho_n = A\alpha^n + B\beta^n + C\gamma^n \qquad \text{(all } n\text{)},$$

where A, B, C are determined by

$$\frac{1}{1 - vx + ux^2 - x^3} = \frac{A}{1-\beta\gamma x} + \frac{B}{1-\gamma\alpha x} + \frac{C}{1-\alpha\beta x}$$

Thus

(1.9) $$\sum_{n=0}^{\infty} \rho_{-n}x^n = \frac{1}{1 - vx + ux^2 - x^3}$$

and

(1.10) $$\sum_{n=0}^{\infty} \rho_n x^n = \frac{1 - ux + vx^2}{1 - ux + vx^2 - x^3} ,$$

while

(1.11) $$\sum_{n=0}^{\infty} \sigma_n x^n = \frac{3 - 2ux + vx^2}{1 - ux + vx^2 - x^3}$$

and

(1.12) $$\sum_{n=0}^{\infty} \sigma_{-n} x^n = \frac{3 - 2vx + ux^2}{1 - vx + ux^2 - x^3}$$

Since $\alpha^3 - \alpha^2 u + \alpha v - 1 = 0$, it is clear from the definition of σ_n, ρ_n that

$$\sigma_{n+3} - u\sigma_{n+2} + v\sigma_{n+1} - \sigma_n = 0$$

and

$$\rho_{n+3} - u\rho_{n+3} + v\rho_{n+1} - \rho_n = 0$$

for arbitrary n.

If we use the fuller notation

$$\sigma_n = \sigma_n(u,v), \qquad \rho_n = \rho_n(u,v),$$

it follows from the generating functions that

(1.13) $$\sigma_{-n}(u,v) = \sigma_n(v,u), \qquad \rho_n(u,v) = \rho_{3-n}(v,u).$$

We show that

(1.14) $$\sigma_m\sigma_n = \sigma_{m+n} + \sigma_{m-n}\sigma_{-n} - \sigma_{m-2n} ,$$

for arbitrary m,n. Similarly

(1.15) $$\sigma_m\rho_n = \rho_{m+n} + \rho_{m-n}\alpha_{-n} - \rho_{m-2n} .$$

As for the product $\rho_m\rho_n$, we have first

(1.16) $$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{2n-6} - \rho_{n-3}\sigma_{n-3} .$$

The more general result is

(1.17) $$2\rho_m\rho_n - \rho_{m+1}\rho_{n-1} - \rho_{m-1}\rho_{n+1}$$

$$= \sigma_{m-3}\sigma_{n-3} - \sigma_{m+n-6} - \sigma_{m-3}\rho_{n-3} - \sigma_{n-3}\rho_{m-3} + 2\rho_{m+n-6} ,$$

again for arbitrary m,n.

Each of the functions $\sigma_n(u,v)$, $\sigma_{-n}(u,v)$, $\rho_n(u,v)$, $\rho_{-n}(u,v)$, $n \geqslant 0$, is a polynomial in u,v. Explicit formulas for these polynomials are given in (2.9), (2.10), (4.5), (4.6) below. Moreover σ_{pn} is a polynomial in σ_n, σ_{-n}; indeed we have

(1.18) $$\sigma_{pn}(u,v) = \sigma_p(\sigma_n, \sigma_{-n}) \qquad (p \geqslant 0).$$

The corresponding formula for ρ_{pn} is somewhat more elaborate; see (4.3) and (4.4) below.

Substitution of the explicit formulas for σ_n, σ_{-n}, ρ_n, ρ_{-n} in any of the relations such as (1.14), (1.15), (1.16), (1.17) gives rise to a large number of polynomial identities.

The introduction of two indeterminates u, v in σ_n, ρ_n leads to somewhat more elaborate formulas than those in [1]. However the greater symmetry implied by (1.13) is gratifying.

2. It follows from

$$1 - ux + vx^2 - x^3 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x) \tag{2.1}$$

that

$$\begin{cases} \alpha + \beta + \gamma = u \\ \beta\nu + \gamma\alpha + \alpha\beta = v \\ \alpha\beta\gamma = 1 \end{cases} \tag{2.2}$$

Since $\alpha\beta\nu = 1$, (2.1) is equivalent to

$$1 - vx + ux^2 - x^3 = (1 - \beta\gamma x)(1 - \gamma\alpha x)(1 - \alpha\beta x). \tag{2.3}$$

We have defined

$$\sigma_n = \alpha^n + \beta^n + \gamma^n , \tag{2.4}$$

for n an arbitrary integer. Thus

$$\sum_{n=0}^{\infty} \sigma_n x^n = \sum \frac{1}{1 - \alpha x} = \frac{\Sigma (1 - \beta x)(1 - \gamma x)}{1 - ux + vx^2 - x^3} ,$$

which, by (2.2), reduces to

$$\sum_{n=0}^{\infty} \sigma_n x^n = \frac{3 - 2ux + vx^2}{1 - ux + vx^2 - x^3} . \tag{2.5}$$

Similarly

$$\sum_{n=0}^{\infty} \sigma_{-n} x^n = \sum \frac{1}{1 - \beta\gamma x} = \frac{(1 - \alpha\beta x)(1 - \alpha\gamma x)}{1 - vx + ux^2 - x^3} .$$

so that

$$\sum_{n=0}^{\infty} \sigma_{-n} x^n = \frac{3 - 2vx + ux^2}{1 - vx + ux^2 - x^3} . \tag{2.6}$$

Using the fuller notation

$$\sigma_n = \sigma_n(u,v), \quad \sigma_{-n} = \sigma_{-n}(u,v),$$

it is clear from (2.5) and (2.6) that

$$\sigma_{-n}(u,v) = \sigma_n(v,u). \tag{2.7}$$

By (2.1), α, β, ν are the roots of

$$z^3 - uz^2 + vz - 1 = 0$$

and so

$$\sigma_{n+3} - u\sigma_{n+2} + v\sigma_{n+1} - \sigma_n = 0, \tag{2.8}$$

for all n.

Next,

$$(1 - ux + vx^2 - x^3)^{-1} = \sum_{k=0}^{\infty} (ux - vx^2 + x^3)^k = \sum_{i,j,k=0}^{\infty} (-1)^j (i,j,k) u^i v^j x^{i+2j+3k}$$

$$= \sum_{n=0}^{\infty} x^n \sum_{i+2j+3k=n} (-1)^j (i,j,k) u^i v^j ,$$

where

$$(i,j,k) = \frac{(i+j+k)!}{i!\,j!\,k!} .$$

Thus, by (2.5),

$$\sigma_n = 3 \sum_{i+2j+3k=n} (-1)^j (i,j,k) u^i v^j - 2u \sum_{i+2j+3k=n-1} (-1)^j (i,j,k) u^i v^j + v \sum_{i+2j+3k=n-2} (-1)^j (i,j,k) u^i v^j$$

$$= \sum_{i+2j+3k=n} (-1)^j u^i v^j \{3(i,j,k) - 2(i-1,j,k) - (i,j-1,k)\} .$$

Hence

(2.9) $$\sigma_n = \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) u^i v^j \qquad (n > 0).$$

By (2.7) the corresponding formula for σ_{-n} is

(2.10) $$\sigma_{-n} = \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) v^i u^j \qquad (n > 0).$$

It follows that, for n prime, coefficients of all terms—except the leading term—in σ_n are divisible by n.

Returning to (2.4), we have

$$\sigma_m \sigma_n = \Sigma \alpha^m \Sigma \alpha^n = \Sigma \alpha^{m+n} + \Sigma \alpha^m (\beta^n + \gamma^n) = \sigma_{m+n} + \Sigma \alpha^{m-n} (\alpha^n \beta^n + \alpha^n \gamma^n)$$
$$= \sigma_{m+n} + \Sigma \alpha^{m-n} (\alpha_{-n} - \beta^n \gamma^n),$$

which gives

(2.11) $$\sigma_m \sigma_n = \sigma_{m+n} + \sigma_{m-n} \sigma_{-n} - \sigma_{m-2n} ,$$

valid for all m, n. Replacing m by $m+2n$, (2.11) becomes

(2.12) $$\sigma_{m+3n} - \sigma_{m+2n} \sigma_n + \sigma_{m+n} \sigma_{-n} - \sigma_m = 0 .$$

For $m = n$, (2.11) reduces to

(2.13) $$\sigma_n^2 = \sigma_{2n} + 2\sigma_{-n} .$$

Hence, for $m = 2n$,

$$\sigma_n \sigma_{2n} = \sigma_{3n} + \sigma_n \sigma_{-n} - 3,$$

so that

(2.14) $$\sigma_{3n} = \sigma_n^3 - 3\sigma_n \sigma_{-n} + 3 .$$

To get the general formula we take

$$\sum_{p=0}^{\infty} \sigma_{pn} x^k = \sum \frac{1}{1-\alpha^n x} = \frac{\Sigma (1-\beta^n x)(1-\gamma^n x)}{(1-\alpha^n x)(1-\beta^n x)(1-\gamma^n x)} = \frac{3 - 2\sigma_n x + \sigma_{-n} x^2}{1 - \sigma_n x + \sigma_{-n} x^2 - x^3} .$$

Comparing with (2.5), it is evident from (2.9) that

(2.15) $$\sigma_{pn} = \sum_{i+2j+3k=p} (-1)^j \frac{p}{i+j+k} (i,j,k) \sigma_n^i \sigma_{-n}^j \qquad (p > 0).$$

Substitution from (2.9) and (2.10) in (2.11), (2.12), (2.13), (2.14), (2.15) evidently results in a number of combinatorial identities. We state only

(2.16) $$\left\{ \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) u^i v^j \right\}^2 = \sum_{i+2j+3k=2n} (-1)^j \frac{2n}{i+j+k} (i,j,k) u^i v^j + 2 \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) v^i u^j$$
$$(n \geq 0).$$

3. Put

(3.1) $$\frac{1}{1-vx+ux^2-x^3} = \frac{A}{1-\beta\gamma x} + \frac{B}{1-\gamma\alpha x} + \frac{C}{1-\alpha\beta x},$$

where A, B, C are independent of x. Then

(3.2) $$(1-\alpha^2\beta)(1-\alpha^2\gamma)A = 1.$$

Since

$$(1-\alpha^2\beta)(1-\alpha^2\gamma) = 1-\alpha^2(\beta+\gamma)+\alpha^4\beta\gamma = 1-\alpha^2(u-\alpha)+\alpha^3 = 1-\alpha^2u+2\alpha^3,$$

it follows from $\alpha^3-\alpha^2u+\alpha v-1=0$ that

(3.3) $$A = \frac{1}{3-2\alpha v+\alpha^2u}$$

with similar formulas for B and C.

Replacing x by $1/x$ in (3.1) and simplifying, we get

$$\frac{x^3}{1-ux+vx^2-x^3} = -\sum\frac{Ax}{\beta\gamma-x} = \sum\frac{A\alpha x}{1-\alpha x} = \sum\frac{A}{1-\alpha x} - \sum A.$$

Since $\Sigma A = 1$, it follows that

(3.4) $$\frac{1-ux+vx^2}{1-ux+vx^2-x^3} = \sum\frac{A}{1-\alpha x}.$$

We now define ρ_n, ρ_{-n} by means of

(3.5) $$\frac{1-ux+vx^2}{1-ux+vx^2-x^3} = \sum_{n=0}^{\infty}\rho_n x^n$$

and

(3.6) $$\frac{1}{1-vx+ux^2-x^3} = \sum_{n=0}^{\infty}\rho_{-n}x^n.$$

It then follows from (3.1) and (3.4) that

(3.7) $$\rho_n = \Sigma A\alpha^n,$$

for all n.

By (3.6), we have, for arbitrary m and n,

$$\rho_m\rho_n = \Sigma A\alpha^m\cdot\Sigma A\alpha^n = \Sigma A^2\alpha^{m+n} + \Sigma BC(\beta^m\gamma^n+\gamma^m\beta^n).$$

Thus

$$\rho_{m+1}\rho_{n-1} = \Sigma A^2\alpha^{m+n} = BC(\beta^{m+1}\gamma^{n-1}+\gamma^{m+1}\beta^{n-1}),$$

so that

(3.8) $$\rho_m\rho_n - \rho_{m+1}\rho_{n-1} = \Sigma BC\left\{(\beta^m\gamma^n+\gamma^m\beta^n)-(\beta^{m+1}\gamma^{n-1}+\gamma^{m+1}\beta^{n-1})\right\}.$$

The quantity in braces is equal to

$$-(\beta-\gamma)(\beta^m\gamma^{n-1}-\gamma^m\beta^{n-1}).$$

Hence

$$\begin{cases}\rho_m\rho_n-\rho_{m+1}\rho_{n-1} = -\Sigma BC(\beta-\gamma)(\beta^m\gamma^{n-1}-\gamma^m\beta^{n-1})\\ \rho_m\rho_n-\rho_{m-1}\rho_{n+1} = -\Sigma BC(\beta-\gamma)(\beta^n\gamma^{m-1}-\gamma^n\beta^{m-1})\end{cases}.$$

It follows that

(3.9) $$2\rho_m\rho_n-\rho_{m+1}\rho_{n-1}-\rho_{m-1}\rho_{n+1} = -\Sigma BC(\beta-\gamma^2(\beta^{m-1}\gamma^{n-1}+\gamma^{m-1}\beta^{n-1}).$$

By (3.2),

$$BC(\beta-\gamma)^2 = -A\alpha^2,$$

so that (3.9) becomes

$$2\rho_m\rho_n - \rho_{m+1}\rho_{n-1} - \rho_{m-1}\rho_{n+1} = \Sigma A(\beta^{m-3}\gamma^{n-3} + \gamma^{m-3}\beta^{n-3}). \tag{3.10}$$

In particular, if $m = n$, (3.10) reduces to

$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \Sigma A\beta^{n-3}\gamma^{n-3} = \Sigma A\alpha^{-n+3}$$

and so

$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{-n+3} \qquad \text{(all } n). \tag{3.11}$$

To get a more general result consider

$$\beta^m\nu^n + \gamma^m\beta^n = (\beta^m + \gamma^m)(\beta^n + \gamma^n) - (\beta^{m+n} + \gamma^{m+n}) = (\sigma_m - \alpha^m)(\sigma_n - \alpha^n) - (\sigma_{m+n} - \alpha^{m+n})$$
$$= \sigma_m\sigma_n - \sigma_m\alpha^n - \sigma_n\alpha^m - \sigma_{m+n} + 2\alpha^{m+n}.$$

Thus

$$\Sigma A(\beta^m\gamma^n + \gamma^m\beta^n) = \sigma_m\sigma_n - \sigma_{m+n} - \sigma_m\beta_n - \sigma_n\beta_m + 2\rho_{m+n}. \tag{3.12}$$

Combining (3.10) and (3.12) we get

$$2\rho_m\rho_n - \rho_{m+1}\rho_{n-1} - \rho_{m-1}\rho_{n+1} = \sigma_{m-3}\sigma_{n-3} - \sigma_{m+n-6} - \sigma_{m-3}\rho_{n-3} - \sigma_{n-3}\rho_{m-3} + 2\rho_{m+n-6}. \tag{3.13}$$

For $m = n$, (3.13) reduces to

$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{2n-6} - \sigma_{n-3}\rho_{n-3} + \sigma_{-n+3}. \tag{3.14}$$

It is not evident that (3.14) is equivalent to (3.11). This is proved immediately below.

4. We now take

$$\rho_m\sigma_n = \Sigma A\alpha^m\Sigma\alpha^n = \Sigma A\alpha^{m+n} + \Sigma A\alpha^m(\beta^n + \gamma^n) = \rho_{m+n} + \Sigma A\alpha^{m-n}(\alpha^n\beta^n - \alpha^n\gamma^n)$$
$$= \rho_{m+n} + \Sigma A\alpha^{m-m}(\sigma_{-n} - \alpha^{-n}),$$

which gives

$$\rho_m\sigma_n = \rho_{m+n} + \rho_{m-n}\sigma_{-n} - \rho_{m-2n}. \tag{4.1}$$

In particular, for $m = n$,

$$\rho_n\sigma_n = \rho_{2n} + \sigma_{-n} - \rho_{-n}, \tag{4.2}$$

which shows that (3.14) is indeed equivalent to (3.11).

For $m = 2n$, (4.1) gives

$$\rho_{3n} = \rho_{2n}\sigma_n - \rho_n\sigma_{-n} + 1 = \rho_n\sigma_n^2 - \sigma_n\sigma_{-n} + \rho_{-n}\sigma_n - \rho_n\sigma_{-n} + 1.$$

To get a general formula for ρ_{pn} take

$$\sum_{p=0}^{\infty}\rho_{pn}x^p = \sum_{p=0}^{\infty}x^p\sum A\alpha^{pn} = \sum\frac{A}{1-\alpha^n x} = \frac{\Sigma A(1-\beta^n x)(1-\gamma^n x)}{(1-\alpha^n x)(1-\beta^n x)(1-\gamma^n x)}$$
$$= \frac{1-(\sigma_n - \rho_n)n + \rho_{-n}x^2}{1-\sigma_n x + \sigma_{-n}x^2 - x^3}.$$

Then, as in the proof of (2.15), we have

$$\rho_{pn} = c_{p,n} - (\sigma_n - \rho_n)c_{p-1,n} + \rho_{-n}c_{p-2,n} \qquad (p \geqslant 0), \tag{4.3}$$

where

$$c_{p,n} = \sum_{i+2j+3k=p}(-1)^j(i,j,k)\sigma_n^i\sigma_{-n}^j. \tag{4.4}$$

Since

$$\rho_1 = \Sigma A\alpha = 0, \qquad \rho_2 = \Sigma A\alpha^2 = 0,$$

we have in particular

(4.5)
$$\rho_p = \sum_{i+2j+k=p-3} (-1)^j (i,j,k) u^i v^j \qquad (p \geq 3)$$

and

(4.6)
$$\rho_{-p} = \sum_{i+2j+3k=p} (-1)^j (i,j,k) v^i u^j \qquad (p \geq 0).$$

With the fuller notation

$$\rho_n = \rho_n(u,v), \qquad \rho_{-n} = \rho_{-n}(u,v),$$

it is clear from (4.5) and (4.6) that

(4.7)
$$\rho_n(u,v) = \rho_{3-n}(v,u).$$

Moreover (4.4) becomes

(4.8)
$$c_{p,n} = \rho_p(\sigma_n, \sigma_{-n}) \qquad (p \geq 0).$$

We may now substitute from the explicit formulas (2.9), (2.10), (4.5), (4.6) in various formulas of Sections 3 and 4 to obtain a large number of polynomial identities in two indeterminants. To give only one relatively simple example, we take (4.2). Thus

(4.9)
$$\left\{ \sum_{i+2j+3k=n-3} (-1)^j (i,j,k) u^i v^j \right\} \left\{ \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) u^i v^j \right\}$$
$$= \sum_{i+2j+3k=2(n-3)} (-1)^j (i,j,k) u^i v^j - \sum_{i+2j+3k=n} (-1)^j (i,j,k) v^i u^j$$
$$+ \sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} (i,j,k) v^i u^j \qquad (n \geq 0).$$

5. For small n, σ_n and ρ_n can be computed without much labor by means of the recurrences. Moreover the results are extended by the symmetry relations

$$\sigma_{-n}(u,v) = \sigma_n(v,u), \qquad \rho_n(u,v) = \rho_{3-n}(v,u).$$

A partial check on σ_n is furnished by the result, that, for prime n,

$$\sigma_n(u,v) \equiv u^n \pmod{n}.$$

Also, by (2.5),

$$\sum_{n=0}^{\infty} \sigma_n(1,1) x^n = \frac{3-2x+x^2}{1-x+x^2-x^3} = \frac{3+x-x^2+x^3}{1-x^4},$$

which implies

$$\sigma_n(1,1) = 3, \quad \sigma_{4n+1}(1,1) = \sigma_{4n+3}(1,1) = 1, \quad \sigma_{4n+2}(1,1) = -1.$$

As for $\rho_n(1,1)$, we have by (3.5)

$$\sum_{n=0}^{\infty} \rho_n(1,1) x^n = \frac{1-x+x^2}{1-x+x^2-x^3} = \frac{1+x^3}{1-x^4},$$

so that

$$\rho_{4n}(1,1) = \rho_{4n+3}(1,1) = 1, \qquad \rho_{4n+1}(1,1) = \rho_{4n+2}(1,1) = 0.$$

Table 1

$\sigma_0 = 3, \quad \sigma_1 = u, \quad \sigma_2 = u^2 - 2v$
$\sigma_3 = u^3 - 3uv + 3$
$\sigma_4 = u^4 - 4u^2v + 2v^2 + 4u$
$\sigma_5 = u^5 - 5u^3v + 5uv^2 + 5u^2 - 5v$
$\sigma_6 = u^6 - 6u^4v + 9u^2v^2 + 6u^3 - 2v^3 + 12uv + 3$
$\sigma_7 = u^7 - 7u^5v + 14u^3v^2 + 7u^4 - 7uv^3 - 21u^2v + 7v^2 + 7u$
$\sigma_8 = u^8 - 8u^6v + 20u^4v^2 + 8u^5 - 16u^2v^3 - 32u^3v + 2v^4 + 24uv^2 + 12u^2 - 8v$
$\sigma_9 = u^9 - 9u^7v + 27u^5v^2 + 9u^6 - 30u^3v^3 - 45u^4v + 9uv^4 + 54u^2v^2 + 18u^3 - 9v^2 - 27uv + 3$
$\sigma_{10} = u^{10} - 10u^8v + 35u^6v^2 + 10u^7 - 50u^4v^3 - 60u^5v + 25u^2v^4 + 100u^3v^2 - 2v^5 + 25u^4 - 40uv^3 - 60u^2v + 15v^2 + 10u$

Table 2

$\rho_0 = 1, \quad \rho_1 = \rho_2 = 0, \quad \rho_3 = 1$
$\rho_4 = u, \quad \rho_5 = u^2 - v$
$\rho_6 = u^3 - 2uv + 1$
$\rho_7 = u^4 - 3u^2v + v^2 + 2u$
$\rho_8 = u^5 - 4u^3v + 3uv^2 + 3u^2 - 2v$
$\rho_9 = u^6 - 5u^4v + 6u^2v^2 + 4u^3 - v^3 - 6uv + 1$
$\rho_{10} = u^7 - 6u^5v + 10u^3v^2 + 5u^4 - 4uv^3 - 12u^2v + 3v^2 + 3u$

REFERENCES

1. L. Bernstein, "Zeros of Combinatorial Functions and Combinatorial Identities," *Houston Journal of Math.*, Vol. 2 (1976), pp. 9–16.
2. L. Carlitz, "Some Combinatorial Identities of Bernstein," *SIAM Journal of Mathematical Analysis*, to appear.

****