IDENTITIES RELATING THE NUMBER OF PARTITIONS INTO AN EVEN AND ODD NUMBER OF PARTS, II

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Definition. If $i \geqslant 0$ and $n \geqslant 1$, let $q_i^e(n)$ be the number of partitions of n into an even number of parts, where each part occurs at most i times. Let $q_i^o(n)$ be the number of partitions of n into an odd number of parts, where each part occurs at most i times. If $i \geqslant 0$, let $q_i^e(0) = 1$ and $q_i^o(0) = 0$.

Definition. If $i \geqslant 0$ and $n \geqslant 0$, let $\Delta_i(n) = q_i^e(n) - q_i^o(n)$.

The purpose of this paper is to determine $\Delta_i(n)$ when i is any odd positive integer. The only cases previously known were $i = 1$, proved by Euler (see [1]), $i = 3$, proved by this writer (see [2]), and $i = 5$ and 7, proved by Alder and Muwafi (see [3]).

Definition. If s, t, u are positive integers with s odd and $1 \leqslant s < t$, and n is an integer, let $f_{s,t,u}(n)$ be the number of partitions of n in which each odd part occurs at most once and is $\not\equiv \pm s \pmod{2t}$ and in which each even part is divisible by $2t$ and occurs $< u$ times.

Theorem. If s, t, u are positive integers with s odd and $1 \leqslant s < t$, and n is an integer, then

$$\Delta_{2tu-1}(n) = (-1)^n \sum_j f_{s,t,u}(n - tj^2 - (t-s)j).$$

Proof.

$$\sum_n \Delta_{2tu-1}(n)x^n = \prod_{\substack{j \\ j \geqslant 1}} \frac{1-x^{2tuj}}{1+x^j} = \prod_{\substack{j \\ j\geqslant 1 \\ 2\nmid j}} (1-x^j) \cdot \prod_{\substack{j \\ j \geqslant 1 \\ 2t \mid j}} (1-x^j)(1+x^j+x^{2j}+\cdots+x^{(u-1)j})$$

$$= \prod_{\substack{j \\ j \geqslant 0}} (1-x^{2tj+s})(1-x^{2tj+2t-s})(1-x^{2tj+2t}) \cdot \prod_{\substack{j \\ j\geqslant 1 \\ 2\nmid j \\ j \not\equiv \pm s \,(\mathrm{mod}\, 2t)}} (1-x^j) \cdot \prod_{\substack{j \\ j\geqslant 1 \\ 2t \mid j}} (1+x^j+x^{2j}+\cdots+x^{(u-1)j})$$

$$= \sum_j (-1)^j x^{tj^2+(t-s)j} \cdot \prod_{\substack{j \\ j\geqslant 1 \\ 2\nmid j \\ j \not\equiv \pm s \,(\mathrm{mod}\, 2t)}} (1-x^j) \cdot \prod_{\substack{j \\ j\geqslant 1 \\ 2t \mid j}} (1+x^j+x^{2j}+\cdots+x^{(u-1)j}),$$

where the last equality follows from Jacobi's identity with $k = t$ and $\ell = t - s$. Since s is odd,

$$tj^2 + (t-s)j \equiv j \pmod 2.$$

Hence, when we substitute $-x$ for x, we obtain

$$\sum_n (-1)^n \Delta_{2tu-1}(n) x^n = \sum_j x^{tj^2+(t-s)j} \cdot \prod_{\substack{j \\ j \geq 1 \\ 2 \nmid j \\ j \not\equiv \pm s \pmod{2t}}} (1+x^j) \cdot \prod_{\substack{j \\ j \geq 1 \\ 2t \mid j}} (1+x^j+x^{2j}+\dots+x^{(u-1)j})$$

$$= \sum_j x^{tj^2+(t-s)j} \cdot \sum_m f_{s,t,u}(m) x^m ,$$

from which the theorem follows immediately.

Corollary 1. If s and t are positive integers with s odd and $1 \leq s < t$, and n is an integer, then

$$\Delta_{2t-1}(n) = (-1)^n \sum_j f_{s,t,1}(n - tj^2 - (t-s)j).$$

Note that $f_{s,t,1}(n)$ is the number of partitions of n into distinct odd parts $\not\equiv \pm s \pmod{2t}$.

Proof. Let $u = 1$ in the theorem.

Letting $s = 1$ and $t = 3$ yields Theorem 1 of [3].

Corollary 2. If $i \geq 2$ and n is an integer, then $(-1)^n \Delta_i(n) \geq 0$.

Proof. For even i, this follows from Theorem 3 of [2]; for odd i, it follows by letting $s = 1$ and $t = (i+1)/2$ in Corollary 1.

Corollary 3. If s and t are positive integers with s odd and $1 \leq s < t$, and n is an integer, then

$$\Delta_{4t-1}(n) = (-1)^n \sum_j f_{s,t,2}(n - tj^2 - (t-s)j).$$

Note that $f_{s,t,2}(n)$ is the number of partitions of n into distinct parts which are either odd but $\not\equiv \pm s \pmod{2t}$ or which are divisible by $2t$.

Proof. Let $u = 2$ in the theorem.

Corollary 4. If u is a positive integer and n is an integer, then

$$\Delta_{4u-1}(n) = (-1)^n \sum_j f_{1,2,u}(n - 2j^2 - j).$$

Note that $f_{1,2,u}(n)$ is the number of partitions of n into parts divisible by 4, where each part occurs $< u$ times.

Proof. Let $s = 1$, $t = 2$ in the theorem.

Letting $u = 1$ yields Theorem 2 of [2] and $u = 2$, Theorem 2 of [3].

REFERENCES

1. Ivan Niven and Herbert S. Zuckerman, *An Introduction to the Theory of Numbers,* 3rd ed., pp. 221–222, Wiley, New York, 1972.
2. Dean R. Hickerson, "Identities Relating the Number of Partitions into an Even and Odd Number of Parts," *J. Combinatorial Theory, Section A* (1973), pp. 351–353.
3. Henry L. Alder and Amin A. Muwafi, "Identities Relating the Number of Partitions into an Even and Odd Number of Parts," *The Fibonacci Quarterly,* Vol. 13, No. 2 (1975), pp. 337–339.

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