Solution by Paul S. Bruckman, Concord, California.

Since $2m \leqslant \log (k\sqrt{5})/\log a < 2m + 1$, it follows that $a^{2m} \leqslant k\sqrt{5} < a^{2m+1}$; hence,

$$a^{2m} - b^{2m} = a^{2m} - (-a^{-1})^{2m} < k\sqrt{5} < a^{2m+1} - (-a^{-1})^{2m+1} = a^{2m+1} - b^{2m+1},$$

i.e.,

$$F_{2m} < k < F_{2m+1}.$$

Since $\{F_n\}_1^\infty$ is a non-decreasing sequence of positive integers, it follows that $F_n \leqslant k$ for $n = 1, 2, \cdots, 2m$, i.e., for $2m$ (distinct) values of n.

Also solved by A. G. Shannon and the Proposer.

Cont. from p. 183

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$$= 0 \qquad \text{elsewhere.}$$

Therefore,

$$d_{ij} = \sum_{k=1}^{n} c_{ik}a_{kj}^T = \sum_{k=1}^{n} \binom{k-1}{i-k}\binom{j-1}{k-1}.$$

The effective limits of this summation are from $k = 1 + [\frac{1}{2}i]$ to min. (i,j). It will be convenient, however, to consider the upper limit to be equal to i; if $i > j$, the extra terms included vanish in any event. Therefore,

$$d_{ij} = \sum_{k=[\frac{1}{2}i]}^{i-1} \binom{k}{i-1-k}\binom{j-1}{k} = \sum_{k=0}^{[\frac{1}{2}(i-1)]} \binom{i-1-k}{k}\binom{j-1}{i-1-k}.$$

For convenience, let $i - 1 = r$, $j - 1 = s$.

Therefore,

$$d_{ij} = \theta_{rs} = \sum_{k=0}^{[\frac{1}{2}r]} \binom{r-k}{k}\binom{s}{r-k};$$

let

$$y = \sum_{r=0}^{\infty} \theta_{rs}x^r.$$

Then

$$y = \sum_{r=0}^{\infty} x^r \sum_{k=0}^{[\frac{1}{2}r]} \binom{r-k}{k}\binom{s}{r-k} = \sum_{k=0}^{\infty}\sum_{r=2k}^{\infty} x^r \binom{r-k}{k}\binom{s}{r-k} = \sum_{k=0}^{\infty} x^{2k} \sum_{r=0}^{\infty} x^r \binom{r+k}{k}\binom{s}{r+k}.$$

Thus,

$$y = \sum_{k=0}^{\infty} \binom{s}{k} x^{2k} \sum_{r=0}^{\infty} \binom{s-k}{r} x^r,$$

by rearranging the combinatorial terms. Then,

$$y = \sum_{k=0}^{\infty} \binom{s}{k} x^{2k}(1+x)^{s-k} = (1+x)^s \sum_{k=0}^{\infty} \binom{s}{k}\left(\frac{x^2}{1+x}\right)^k = (1+x)^s\left(1+\frac{x^2}{1+x}\right)^s,$$

or:

(2) $$y = (1+x+x^2)^s.$$

Therefore, d_{ij} is the coefficient of x^{i-1} in $(1+x+x^2)^{j-1}$. From this, we may deduce that the d_{ij}'s satisfy the following recursion:

(3) $$d_{i+2:j+1} = d_{ij} + d_{i+1:j} + d_{i+2:j}\ (i,j \geqslant 1);\ d_{1:j} = 1,\ d_{2:j} = j-1\ (j \geqslant 1);\ d_{i:1} = 0\ (i > 1).$$

We may readily construct a matrix (of unspecified dimensions), whose j^{th} column is composed of the coefficients of $(1+x+x^2)^{j-1}$, written in correspondence to the ascending powers of x, beginning with x^0. For any given j, $d_{ij} = 0$ for all $i \geqslant 2j$ (since $(1+x+x^2)^{j-1}$ contains $(2j-1)$ non-zero terms).

Also solved by the Proposer.