ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-376 Proposed by Frank Kocher and Gary L. Mullen, Pennsylvania State University, University Park and Sharon, Pennsylvania.

Find all integers $n > 3$ such that $n - p$ is an odd prime for all odd primes p less than n.

B-377 Proposed by Paul S. Bruckman, Concord, California.

For all real numbers $a \geqslant 1$ and $b \geqslant 1$, prove that

$$\sum_{k=1}^{[a]} [b\sqrt{1-(k/a)^2}] = \sum_{k=1}^{[b]} [a\sqrt{1-(k/b)^2}],$$

where $[x]$ is the greatest integer in x.

B-378 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $F_{3n+1} + 4^n F_{n+3} \equiv 0 \pmod 3$ for $n = 0, 1, 2, \cdots$.

B-379 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that $F_{2n} \equiv n(-1)^{n+1} \pmod 5$ for all non-negative integers n.

B-380 Proposed by Dan Zwillinger, Cambridge, MA.

Let a, b, and c be non-negative integers. Prove that

$$\sum_{k=1}^{n} \binom{k+a-1}{a}\binom{n-k+b-c}{b} = \binom{n+a+b-c}{a+b+1}.$$

Here $\binom{m}{r} = 0$ if $m < r$.

B-381 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $a_{2n} = F_{n+1}^2$ and $a_{2n+1} = F_{n+1}F_{n+2}$. Find the rational function that has

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

as its Maclaurin series.

SOLUTIONS

C IS EASY TO SEE

B-352 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let S_n be defined by $S_0 = 1$, $S_1 = 2$, and

$$S_{n+2} = 2S_{n+1} + cS_n .$$

For what value of c is $S_n = 2^n F_{n+1}$ for all non-negative integers n?

Solution by Paul S. Bruckman, Concord, California.

Substituting the definition of S_n into the given recursion yields:

$$2^{n+2}F_{n+3} = 2^{n+2}F_{n+2} + c\cdot 2^n F_{n+1}, \quad \text{or} \quad F_{n+3} = F_{n+2} + \tfrac{1}{4}c\cdot F_{n+1}.$$

Since

$$F_{n+3} = F_{n+2} + F_{n+1},$$

it follows that $c = 4$.

Also solved by George Berzsenyi, Wray G. Brady, Herta T. Freitag, Ralph Garfield, Dinh Thế Hung, John Ivie, Graham Lord, John W. Milsom, C. B. A. Peck, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, David Zeitlin, and the Proposer.

RECURSIVE SUMS

B-353 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

For k and n ontegers with $0 \leqslant k \leqslant n$, let $A(k,n)$ be defined by $A(0,n) = 1 = A(n,n)$, $A(1.2) = c + 2$ and

$$A(k+1, n+2) = cA(k,n) + A(k, n+1) + A(k+1, n+1).$$

Also let $S_n = A(0,n) + A(1,n) + \cdots + A(n,n)$. Show that

$$S_{n+2} = 2S_{n+1} + cS_n .$$

Solution by A. G. Shannon, New South Wales, I. of T., Australia.

$$S_{n+2} = \sum_{i=0}^{n+2} A(i, n+2) = 2 + \sum_{i=1}^{n+1} A(i, n+2) = 2 + c\sum_{i=1}^{n+1} A(i-1, n) + \sum_{i=1}^{n+1} A(i-1, n+1) + \sum_{i=1}^{n+1} A(i, n+1)$$

$$= 2 + c\sum_{i=0}^{n} A(i,n) + \sum_{i=0}^{n} A(i, n+1) + \sum_{i=1}^{n+1} A(i, n+1) = 2\sum_{i=0}^{n+1} A(i, n+1) + c\sum_{i=0}^{n} A(i,n)$$

$$= 2S_{n+1} + cS_n,$$

as required.

Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, Dinh Thế Hung, John Ivie, Graham Lord, John W. Milsom, C. B. A. Peck, Bob Prielipp, David Zeitlin and the Proposer.

A VANISHING FACTOR

B-354 Proposed by Phil Mana, Albuquerque, New Mexico.

Show that

$$F_{n+k}^3 - L_k^3 F_n^3 + (-1)^k F_{n-k}[F_{n-k}^2 + 3F_{n+k}F_n L_k] = 0.$$

Solution by Graham Lord, Universite' Laval, Quëbec, Canada.

This follows from a special case of the algebraic identity

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca),$$

where

$$a = F_{n+k}, \quad b = -L_kF_n \quad \text{and} \quad c = (-1)^kF_{n-k}.$$

Note that

$$F_{n+k} - L_kF_n + (-1)^kF_{n-k} = 0.$$

Also solved by Wray G. Brady, Paul S. Bruckman, Ralph Garfield, Dinh Thế Hung, John W. Milsom, C. B. A. Peck, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, and the Proposer.

CUBIC IDENTITY

B-355 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that

$$F_{n+k}^3 - L_{3k}F_n^3 + (-1)^kF_{n-k}^3 = 3(-1)^nF_nF_kF_{2k}.$$

Solution by Graham Lord, Université Laval, Québec, Canada.

The replacement of L_{3k} by $L_k^3 - 3(-1)^kL_k$ and the utilization of the identity of problem B-354 changes the left-hand side above into

$$3(-1)^kL_kF_n[F_n^2 - F_{n+k}F_{n-k}],$$

which is the same as

$$3(-1)^kL_kF_n(-1)^{n+k+2}F_k^2,$$

that is $3(-1)^nF_nF_kF_{2k}$.

Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, John W. Milsom, C. B. A. Peck, Bob Prielipp, A. G. Shannon, and the Proposer.

SOME SOLUTIONS

B-356 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let $S_n = F_2 + 2F_4 + 3F_6 + \cdots + nF_{2n}$. Find m as a function of n so that F_{m+1} is an integral divisor of $F_m + S_n$.

Solution by Paul S. Bruckman, Concord, California.

We first find a closed expression for S_n. Note that

$$S_n = \sum_{k=1}^{n} kF_{2k} = \sum_{k=1}^{n}\left\{kF_{2k+1} - (k-1)F_{2k-1} - F_{2k} + F_{2k-2}\right\} = \left\{(k-1)F_{2k-1} - F_{2k-2}\right\}\Big|_{k=1}^{n+1}$$

$$= nF_{2n+1} - F_{2n}.$$

Clearly,

$$F_{2n} + S_n = nF_{2n+1},$$

and so $m = 2n$ is a solution of the problem. Since $F_1 = F_2 = 1$, it is clear that $m = 0$ and $m = 1$ are also (trivial) solutions. The statement of the problem seems to require finding *all* solutions m, and this appears to be a difficult task, perhaps not intended by the Proposer.

Also solved by George Berzsenyi, Wray G. Brady, Graham Lord, Bob Prielipp, A. G. Shannon, Gregory Wulczyn, David Zeitlin and the Proposer.

GOLDEN RATION INEQUALITY COUNT

B-357 Proposed by Frank Higgins, Naperville, Illinois.

Let m be a fixed positive integer and let k be a real number such that

$$2m \leqslant \frac{\log(\sqrt{5}k)}{\log a} < 2m+1,$$

where $a = (1+\sqrt{5})/2$. For how many positive integers n is $F_n \leqslant k$?

Solution by Paul S. Bruckman, Concord, California.

Since $2m \leqslant \log(k\sqrt{5})/\log a < 2m+1$, it follows that $a^{2m} \leqslant k\sqrt{5} < a^{2m+1}$; hence,

$$a^{2m} - b^{2m} = a^{2m} - (-a^{-1})^{2m} < k\sqrt{5} < a^{2m+1} - (-a^{-1})^{2m+1} = a^{2m+1} - b^{2m+1},$$

i.e.,

$$F_{2m} < k < F_{2m+1}.$$

Since $\{F_n\}_1^\infty$ is a non-decreasing sequence of positive integers, it follows that $F_n \leqslant k$ for $n = 1, 2, \cdots, 2m$, i.e., for $2m$ (distinct) values of n.

Also solved by A. G. Shannon and the Proposer.

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$$= 0 \qquad \text{elsewhere.}$$

Therefore,

$$d_{ij} = \sum_{k=1}^{n} c_{ik} a_{kj}^T = \sum_{k=1}^{n} \binom{k-1}{i-k}\binom{j-1}{k-1}.$$

The effective limits of this summation are from $k = 1 + [\tfrac{1}{2}i]$ to min. (i,j). It will be convenient, however, to consider the upper limit to be equal to i; if $i > j$, the extra terms included vanish in any event. Therefore,

$$d_{ij} = \sum_{k=[\frac{1}{2}i]}^{i-1} \binom{k}{i-1-k}\binom{j-1}{k} = \sum_{k=0}^{[\frac{1}{2}(i-1)]} \binom{i-1-k}{k}\binom{j-1}{i-1-k}.$$

For convenience, let $i - 1 = r$, $j - 1 = s$.

Therefore,

$$d_{ij} = \theta_{rs} = \sum_{k=0}^{[\frac{1}{2}r]} \binom{r-k}{k}\binom{s}{r-k};$$

let

$$y = \sum_{r=0}^{\infty} \theta_{rs} x^r.$$

Then

$$y = \sum_{r=0}^{\infty} x^r \sum_{k=0}^{[\frac{1}{2}r]} \binom{r-k}{k}\binom{s}{r-k} = \sum_{k=0}^{\infty}\sum_{r=2k}^{\infty} x^r \binom{r-k}{k}\binom{s}{r-k} = \sum_{k=0}^{\infty} x^{2k} \sum_{r=0}^{\infty} x^r \binom{r+k}{k}\binom{s}{r+k}.$$

Thus,

$$y = \sum_{k=0}^{\infty} \binom{s}{k} x^{2k} \sum_{r=0}^{\infty} \binom{s-k}{r} x^r,$$

by rearranging the combinatorial terms. Then,

$$y = \sum_{k=0}^{\infty} \binom{s}{k} x^{2k}(1+x)^{s-k} = (1+x)^s \sum_{k=0}^{\infty} \binom{s}{k}\left(\frac{x^2}{1+x}\right)^k = (1+x)^s\left(1+\frac{x^2}{1+x}\right)^s,$$

or:

(2) $$y = (1+x+x^2)^s.$$

Therefore, d_{ij} is the coefficient of x^{i-1} in $(1+x+x^2)^{j-1}$. From this, we may deduce that the d_{ij}'s satisfy the following recursion:

(3) $$d_{i+2:j+1} = d_{ij} + d_{i+1:j} + d_{i+2:j} \;(i,j \geqslant 1);\; d_{1:j} = 1,\; d_{2:j} = j-1 \;(j \geqslant 1);\; d_{i:1} = 0 \;(i > 1).$$

We may readily construct a matrix (of unspecified dimensions), whose j^{th} column is composed of the coefficients of $(1+x+x^2)^{j-1}$, written in correspondence to the ascending powers of x, beginning with x^0. For any given j, $d_{ij} = 0$ for all $i \geqslant 2j$ (since $(1+x+x^2)^{j-1}$ contains $(2j-1)$ non-zero terms).

Also solved by the Proposer.