GEOMETRIC SEQUENCES AND THE INITIAL DIGIT PROBLEM

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Recently, R. L. Duncan discussed the initial digit problem for the sequence of positive integers, J [1]. The subsequence of positive integers with initial digit $a \in \{1, 2, \cdots, 9\}$ is denoted by $A = \{a_\gamma\}$. Although the asymptotic density of A in J does not exist, the logarithmic density of A in J is $\log(1 + 1/a)$, where $\log x$ is the common logarithm of x.

The purpose of this note is to show that the relative asymptotic density of A in certain geometric sequences is also $\log(1 + 1/a)$.

Let c denote a positive integer which is not a power of ten. We adopt the following definitions.

Definition 1. $$B(m) = \{y \mid y = c^n,\ n \geqslant 1,\ y \leqslant c^m,\ m \in J\}.$$

Definition 2. $$B' = \bigcup_{m \in J} B(m).$$

Definition 3. $$A(m) = A \cap B(m).$$

Definition 4. $$A' = A \cap B'.$$

Definition 5. $$a(m) = \sum_{y \in A(m)} 1.$$

Definition 6. $$b(m) = \sum_{y \in B(m)} 1 = m.$$

Clearly $c^m \in A'$ iff

(1) $$a10^t \leqslant c^m < (a+1)10^t \qquad (t \geqslant 0).$$

But (1) is equivalent to

(2) $$\left[\frac{t + \log a}{\log c} \leqslant m < \frac{t + \log(a+1)}{\log c}\right).$$

Let

$$I_{t+1} = \left[\frac{t + \log a}{\log c}, \frac{t + \log(a+1)}{\log c}\right) \qquad (t \geqslant 0)$$

and $|I_{t+1}|$ denote the length of I_{t+1}.

Obviously

(3) $$|I_{t+1}| = \frac{\log(1 + 1/a)}{\log c} \leqslant \frac{\log 2}{\log c} \leqslant 1.$$

In fact, $|I_{t+1}| = 1$ iff $a = 1$ and $c = 2$.

Let z_{t+1} denote the midpoint of I_{t+1}.

(4) $$z_{t+1} = \frac{2t + \log a(a+1)}{\log c} \qquad (t \geqslant 0).$$

Lemma 1. $\{z_t\}_{t=1}^{\infty}$ is uniformly distributed mod 1.

Proof. $$\lim_{t \to \infty}(z_{t+1} - z_t) = \lim_{t \to \infty} \frac{2}{\log c} = \frac{2}{\log c}$$ and $\frac{2}{\log c}$ is irrational [2].

Hence, $\{z_t\}_{t=1}^{\infty}$ is uniformly distributed mod 1 [3].

Lemma 2.

$$a\left(\left[\frac{n+\log(a+1)}{\log c}\right]\right) = |I_1|n + o(n),$$

where $[x]$ denotes the greatest integer in x.

Proof. Obviously

$$a\left(\left[\frac{n+\log(a+1)}{\log c}\right]\right)$$

is the number of intervals, $I_1, I_2, \cdots, I_n$ which contain an integer and this is n less the number of intervals which contain no integer. Since $|I_{t+1}| \leqslant 1$, it is clear that each interval contains at most one integer. If $|I_{t+1}| = 1$ ($c = 2, a = 1$), then

$$a\left(\left[\frac{n+\log 2}{\log 2}\right]\right) = n = |I_1|n + o(n).$$

If $|I_{t+1}| < 1$, I_{t+1} will not contain an integer if, and only if

$$z_{t+1} \in \left(j + \frac{|I_1|}{2},\ j + 1 - \frac{|I_1|}{2}\right),$$

where $z_{t+1} \in (j, j+1)$ for some integer, j. Using Lemma 1 and the definition of uniform distribution mod 1 [4], we have

$$n - a\left(\left[\frac{n+\log(a+1)}{\log c}\right]\right) = (1 - |I_1|)n + o(n)$$

and the result follows.

Let $d(a)$ denote the relative asymptotic density of A' in B' defined as follows:

$$d(a) = \lim_{x\to\infty} \sum_{\substack{a\gamma \leqslant x \\ a\gamma \in A'}} 1 \Big/ \sum_{\substack{n \leqslant x \\ n \in B'}} 1 . \tag{5}$$

The upper and lower relative asymptotic densities of A' in B' are obtained by replacing "limit" in (6) by "limit superior" and "limit inferior," respectively, and are denoted by $\overline{d(a)}$ and $\underline{d(a)}$, respectively [5]. We conclude the discussion with our main result.

Theorem.

$$d(a) = \log(1 + 1/a).$$

Proof. It is clear that

$$\underline{d(a)} = \lim_{n\to\infty} \frac{a\left(\left[\frac{n+\log(a+1)}{\log c}\right]\right)}{b\left(\left[\frac{n+1+\log(a+1)}{\log c}\right]\right) - 1} \tag{6}$$

$$\overline{d(a)} = \lim_{n\to\infty} \frac{a\left(\left[\frac{n+\log(a+1)}{\log c}\right]\right)}{b\left(\left[\frac{n+\log(a+1)}{\log c}\right]\right)}. \tag{7}$$

Since

$$\left[\frac{n+\log(a+1)}{\log c}\right] \sim \frac{n+\log(a+1)}{\log c},$$

the application of Lemma 2 transforms (6) and (7) into

$$\underline{d(a)} = \lim_{n\to\infty} \frac{|I_1|n + o(n)}{\frac{n+1+\log(a+1)}{\log c} + o(n)} = |I_1|\log c = \log(1 + 1/a) \tag{8}$$

and

(9) $$\overline{d(a)} = \lim_{n\to\infty} \frac{|I_1|n + o(n)}{\dfrac{n + \log(a+1)}{\log c} + o(n)} = \log(1 + 1/a)$$

and the desired conclusion follows.

REFERENCES

1. R. L. Duncan, "Note on the Initial Digit Problem," *The Fibonacci Quarterly,* Vol. 7, No. 5, pp. 474–475.
2. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* 4th ed., 1960, p. 46.
3. J. G. Van der Corput, "Diophantische Ungleichungen I: Zur Gleich Verteilung Modulo Eins," *Acta Math.,* 1930-31 (378), pp. 55–56.
4. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* 4th ed., 1960, p. 390.
5. H. Halberstam and K. F. Roth, *Sequences,* Oxford, 1966, Vol. I.

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ADDENDA TO ADVANCED PROBLEMS AND SOLUTIONS

These problem solutions were inadvertently skipped over for a few years. Our apologies.

FORM TO THE RIGHT

H-211 Proposed by S. Krishman, Orissa, India. (corrected)

A. Show that $\binom{2n}{n}$ is of the form $2n^3k + 2$ when n is prime and $n > 3$.

B. Show that $\binom{2n-2}{n-1}$ is of the form $n^3k - 2n^2 - n$, when n is prime.

$\binom{m}{j}$ represents the binomial coefficient, $\dfrac{m!}{j!(m-j)!}$.

Solution by P. Tracy, Liverpool, New York.

A. The Vandermonde convolution identity is $\binom{n}{m} = \Sigma\binom{n-L}{k}\binom{L}{m-k}$. Appling this to $\binom{2p}{p}$ (using $L = p$), we get

$$\binom{2p}{p} = \sum_{k=0}^{p}\binom{p}{k}^2 = 2 + \sum_{k=1}^{p-1}\binom{p}{k}^2 .$$

Since p is a prime, $p \mid \binom{p}{k}$ for $k = 1, 2, \cdots, p-1$. Now

$$\binom{p}{k}^2 \equiv p^2 \left(\frac{(p-1)(p-2)\cdots(p-k+1)}{k!}\right)^2 \pmod{p^3}.$$

Also $(p-i)/i \equiv -1 \pmod{p}$ and so

$$\frac{1}{p^2}\sum_{k=1}^{p-1}\binom{p}{k}^2 \equiv \sum_{k=1}^{p-1}\frac{1}{k^2} \equiv 2 \qquad \text{quad. res.} \pmod{p}$$

(since every quadratic residue mod p has exactly two roots, $\pm a$). Let g be a primitive root, mod p, then the quadratic residues are

$$1, g^2, g^4, \cdots, g^{\frac{p-3}{2}}$$

To find the sum of the quadratic residues, we use the geometric sum formula to obtain $(g^{p-1} - 1)/(g^2 - 1)$. Note that $p > 3$ implies $g^2 - 1 \not\equiv 0 \pmod{p}$. Hence Σ quad. res. $\equiv 0 \pmod{p}$. Therefore

$$2p^3 \,\Big|\, \sum_{k=1}^{p-1}\binom{p}{k}^2 \quad \text{and} \quad \binom{2p}{p} \equiv 2 \pmod{2p^3}.$$

[Continued on page 165.]