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AN INEQUALITY FOR A CLASS OF POLYNOMIALS

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1. INTRODUCTION

Recently, Klamkin and Newman [1], using double induction, proved that

$$\sum_{k=1}^{n} A_k^3 \leqslant \left(\sum_{k=1}^{n} A_k\right)^2 \qquad (n = 1, 2, \cdots), \tag{1.1}$$

where A_k is a non-decreasing sequence with $A_0 = 0$ and $A_k - A_{k-1} \leqslant 1$. For $A_k = k$, (1.1) gives the well known elementary identity

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \qquad (n = 1, 2, \cdots). \tag{1.2}$$

Our inequality (2.1) for polynomials in a single variable x gives (1.1) for $x = 1$.

2. A POLYNOMIAL INEQUALITY

Our first general result is given by

Theorem 1. Let C_k be a non-decreasing sequence with $C_0 = 0$ and $C_k - BC_{k-1} \leqslant 1$, $k = 1, 2, \cdots$, where B is a constant, $0 \leqslant B \leqslant 1$. Then, for $x \geqslant 1$, we have the inequality

$$\sum_{k=1}^{n} C_k^3 x^k \leqslant \left(\sum_{k=1}^{n} C_k x^k\right)^2 \qquad (n = 1, 2, \cdots). \tag{2.1}$$

Proof. We will use double induction. For $n = 1$, (2.1) requires that $C_1^3 x \leqslant C_1^2 x^2$, or $C_1^2 x(C_1 - x) \leqslant 0$, which is true, since $C_1 \leqslant 1$ and $x \geqslant 1$. Assuming (2.1) is true for $k = 1, 2, \cdots, n$, we must now show that

$$\sum_{k=1}^{n+1} C_k^3 x^k = C_{n+1}^3 x^{n+1} + \sum_{k=1}^{n} C_k^3 x^k \leqslant C_{n+1}^3 x^{n+1} + \left(\sum_{k=1}^{n} C_k x^k\right)^2 \leqslant \left(\sum_{k=1}^{n+1} C_k x^k\right)^2,$$

which requires the truth of

$$2\sum_{k=1}^{n} C_k x^k \geqslant C_{n+1}^2 - C_{n+1}x^{n+1} \qquad (n = 1, 2, \cdots). \tag{2.2}$$

For $n = 1$, (2.2) gives

$$C_2^2 - C_2 x^2 \leqslant 2C_1 x.$$

Since $x \geqslant 1$, $x^2 C_2 \geqslant C_2$,

$$C_2^2 - C_2 x^2 \leqslant C_2^2 - C_2;$$

but $C_2 - BC_1 \leqslant 1$, and so

$$C_2^2 - C_2 \leqslant C_1 B C_2 \leqslant C_1 B(1 + BC_1) \leqslant C_1 B(1 + B) \leqslant 2C_1 \leqslant 2C_1 x,$$

which is true since $B(1 + B) \leqslant 2$ for $0 \leqslant B \leqslant 1$. Assuming (2.2) is true for $k = 1, 2, \cdots, n$, we must show that

$$2\sum_{k=1}^{n+1} C_k x^k \geqslant 2C_{n+1}x^{n+1} + (C_{n+1}^2 - C_{n+1}x^{n+1}) \geqslant C_{n+2}^2 - C_{n+2}x^{n+2},$$

which requires that

$$x^{n+1}(xC_{n+2} + C_{n+1}) \geqslant C_{n+2}^2 - C_{n+1}^2, \qquad n = 1, 2, \cdots.$$

Since $B \leqslant 1$, $-BC_{n+1} \geqslant -C_{n+1}$, and so

$$C_{n+2} - C_{n+1} \leqslant C_{n+2} - BC_{n+1} \leqslant 1.$$

Hence

$$C_{n+2}^2 - C_{n+1}^2 \leqslant C_{n+2} + C_{n+1} \leqslant x^{n+1}(xC_{n+2} + C_{n+1}),$$

since $x \geqslant 1$. Thus, the truth of (2.2) completes the proof of Theorem 1.

In [1, p. 29], the following,

Lemma. If $x, y \geqslant 0$, $p \geqslant 2$, then $p(x - y)(x^{p-1} + y^{p-1}) \geqslant 2(x^p - y^p)$,

was used to generalize (1.1) (see [1, (18), p. 29]). Using the above lemma and double induction, we now obtain a generalization of Theorem 1, i.e.,

Theorem 2. Let C_k be a non-decreasing sequence with $C_0 = 0$ and $C_k - BC_{k-1} \leqslant 1$, $k = 1, 2, \cdots$, where B is a constant, $0 \leqslant B \leqslant 1$. Then, for $x \geqslant 1$ and $p = 2, 3, \cdots$, we have the polynomial inequality

$$2\sum_{k=1}^{n} C_k^{2p-1} x^k \leqslant p\left(\sum_{k=1}^{n} C_k^{p-1} x^k\right)^2 \qquad (n = 1, 2, \cdots). \tag{2.3}$$

Remarks. For $p = 2$, (2.3) gives (2.1). For $B = 1$ and $x = 1$, (2.3) gives (18) of [1, p. 29], and (2.1) gives (1.1). The proof of Theorem 2, similar to the proof of Theorem 1, is omitted. We note that when $C_k - BC_{k-1} = 1$ for $k = 1, 2, \cdots$, then

$$C_k = (1 - B^k)/(1 - B),$$

$B \neq 1$ and $C_k = k$ for $B = 1$. For $B = 0$ and $C_k = 1$, $k = 1, 2, \cdots$, (2.1) gives

$$1 \leqslant \sum_{k=1}^{n} x^k$$

so that for $n = 1$, $1 \leqslant x$, as required.

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